5. Equation of state

- **Stellar material is an almost perfect gas.**
- **Main differences from laboratory gas:**
  - **Ionized: plasma allows greater compression (10^{-15} m cf. 10^{-10} m).**
  - **In TE with radiation =>intensity follows Planck's law.**
  - **Particles may not be classical: quantum mechanical effects.**
  - **Particles may be relativistic: special relativity.**
- **A complete description of the macroscopic properties of the gas requires 3 *state variables.***
- **First Law of Thermodynamics:**

$$dE = TdS - PdV + \mu dN$$

Change in internal energy

Temperature

Pressure

Chemical potential (describes effect of a change in number density, e.g. ionization: $H^+ + e^- \rightarrow H^0 + \gamma$)

Spin and polarization

- **Need to allow for intrinsic angular momentum or spin of particles, or for different polarizations of photons:**

$$g(p)dp = g_s \frac{V}{h^3} 4\pi p^2 dp$$

Partition function

Particle type	Spin	g_s	Comment
p, n, e	1/2	2	
ν	1/2	1	Only 1 polarization
photons	1	2	2 independent polarizations for EM wave.

5.2 Occupation probabilities and Internal energy

- **In TE, macroscopic state variables T, P, μ determine equilibrium distribution of particles in quantum states. Different for fermions, bosons.**
- **Energy of particle, mass m, in quantum state with momentum p, is:** $\varepsilon_p^2 = p^2c^2 + m^2c^4$
- **Occupation probability $f(\varepsilon_p)$ is the average number of particles in a state with energy ε_p :**

$$\Rightarrow \text{Internal energy} \quad E = \int_0^\infty \varepsilon_p \, f(\varepsilon_p) \, g(p)dp.$$

$$\text{Total number of particles in gas } N = \int_0^\infty f(\varepsilon_p) \, g(p)dp.$$

5.3 Pressure in an ideal gas – 1

$$P = -\left.\frac{\partial E}{\partial V}\right|_{N,S} = -\int_0^\infty \frac{\partial \varepsilon_p}{\partial V} f(\varepsilon_p) g(p)dp.$$

Use $\dfrac{\partial \varepsilon_p}{\partial V} = \dfrac{\partial \varepsilon_p}{\partial p}\dfrac{\partial p}{\partial V}$. Since $V = L^3$ and $p \propto L$, get

$$p \propto V^{-1/3} \Rightarrow \frac{dp}{dV} = -\frac{p}{3V}.$$

Also $\varepsilon_p^2 = p^2c^2 + m^2c^4 \Rightarrow 2\varepsilon_p \dfrac{\partial \varepsilon_p}{\partial p} = 2pc^2$

$$\Rightarrow \frac{\partial \varepsilon_p}{\partial p} = \frac{pc^2}{\varepsilon_p} \equiv v_p, \text{ so : } \frac{\partial \varepsilon_p}{\partial V} = -\frac{pv_p}{3V}.$$

Speed of particle with momentum p

Pressure in an ideal gas – 2

Hence $P = \frac{1}{3V}\int_0^\infty p v_p f(\varepsilon_p) g(p)\, dp = \frac{N}{3V}\langle p v_p \rangle.$

Average over N particles in gas

Non-relativistic: $\varepsilon_p = mc^2 + \frac{p^2}{2m}$, and $v_p = \frac{p}{m}$

$\Rightarrow P = \frac{2N}{3V}\left\langle \frac{p^2}{2m} \right\rangle =$ 2/3 of kinetic energy density.

Ultra-relativistic: $\varepsilon_p = pc$, and $v_p = c$

$\Rightarrow P = \frac{N}{3V}\langle pc \rangle =$ 1/3 of kinetic energy density.

5.4 Equation of state for an ideal classical gas – 1

$$P = \frac{1}{3V}\int_0^\infty p v_p f(\varepsilon_p) g(p)\, dp$$

Substitute for density of states:

$$g(p)\,dp = g_s \frac{V}{h^3} 4\pi p^2 dp$$

Occupation probability in classical limit:

$$f(\varepsilon_p) \approx \frac{1}{\exp[(\varepsilon_p - \mu)/kT]} << 1$$

Equation of state for an ideal classical gas – 2

$$P = \frac{1}{3V}\exp\left(\frac{\mu}{kT}\right)\int_0^\infty p v_p \exp\left(\frac{-\varepsilon_p}{kT}\right) g_s \frac{V}{h^3} 4\pi p^2 dp.$$

Now $d\varepsilon_p = v_p dp$, so can rewrite integral:

$$\int_0^\infty p^3 \exp\left(\frac{-\varepsilon_p}{kT}\right) v_p dp = -kT\int_0^\infty p^3 d\left(\exp(-\varepsilon_p / kT)\right)$$

Or by parts: $$= 3kT\int_0^\infty \exp(-\varepsilon_p / kT) p^2 dp.$$

Equation of state for an ideal classical gas – 3

- **Substitute back to get expression for the pressure in a classical ideal gas:**

$$P = \frac{kT}{V}\exp\left(\frac{\mu}{kT}\right)\int_0^\infty \exp\left(\frac{-\varepsilon_p}{kT}\right) g_s \frac{V}{h^3} 4\pi p^2 dp.$$

- **Comparing this with expression for the total number of particles:**

$$N = \exp\left(\frac{\mu}{kT}\right)\int_0^\infty \exp\left(\frac{-\varepsilon_p}{kT}\right) g_s \frac{V}{h^3} 4\pi p^2 dp$$

- **leads to the equation of state for an ideal classical gas:**

$$P = \frac{N}{V}kT = nkT.$$

Condition for ideal classical gas

- **In classical limit:**

$$\exp[(mc^2 - \mu)/kT] >> 1.$$

- **Use this to derive an explicit expression for the chemical potential of a classical gas. In equation for total no. of particles substitute:**

$$\varepsilon_p = mc^2 + \frac{p^2}{m} \text{ and integrate to get :}$$

$$N = \exp\left(\frac{\mu - mc^2}{kT}\right) g_s \frac{V}{h^3} (2\pi mkT)^{3/2}.$$

- **Rearrange:**

$$\mu - mc^2 = -kT \ln\left(\frac{g_s n_Q}{n}\right)$$

$$n_Q = \left[\frac{2\pi mkT}{h^2}\right]^{3/2}$$ **Quantum concentration**

$$n = \frac{N}{V}$$

Ultra-relativistic classical gas

- **For UR particles neglect rest energy mc²:**

Substitute $\varepsilon_p = pc$ into expression for N to get :

$$\mu = -kT \ln\left[\frac{g_s n_Q}{n}\right] \text{ where } n_Q = 8\pi \left[\frac{kT}{hc}\right]^3.$$

- **Hence condition for classical gas**

$$\exp\left[(mc^2 - \mu)/kT\right] >> 1$$

is satisfied if n << n_Q.

Fermi-Dirac distribution

- **Total number of particles:**

$$N = \int_0^{p_F} g_s \frac{V}{h^3} 4\pi p^2 dp = \frac{8\pi V}{3h^3} p_F^3 \text{ where } g_s = 2.$$

- **Rearrange to get Fermi momentum:**

$$p_F = h\left[\frac{3n}{8\pi}\right]^{1/3} \text{ where } n = \frac{N}{V} \text{ as usual.}$$

- **To get equation of state evaluate internal energy. Non-relativistic case: p_F << mc implies n << $(mc/h)^3$ ($= (1/\lambda_c)^3$). Subst into expression for E to get:**

$$E = \int_0^{p_F} \varepsilon_p g_s \frac{V}{h^3} 4\pi p^2 dp = N\left[mc^2 + \frac{3p_F^2}{10m}\right].$$

Equation of state

- **NR degenerate electron gas: neglect mc^2 and recall that P = (2/3)E:**

$$P = K_{NR} n^{5/3}, \text{ where } K_{NR} = \frac{h^3}{5m}\left[\frac{3}{8\pi}\right]^{2/3}$$

- **UR degenerate electron gas:**

$$P = K_{UR} n^{4/3}, \text{ where } K_{UR} = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}.$$

- **Note that in both cases, pressure depends only on density, not on temperature.**

5.6 Photon gas

- **Thermal radiation may be described as gas of zero-mass bosons with zero chemical potential.**
- **Photon number density:**
  $n = bT^3$

  where $b = 2.404\ 8\pi(k/hc)^3 = 2.03 \times 10^7 \mathrm{K}^{-3}\ \mathrm{m}^{-3}$.

- **Internal energy density:**
  $U = aT^4$

  where $a = 2.404\ 8\pi^5 k^4 / 15(hc)^3 = 7.565 \times 10^{-16}\ \mathrm{J\ K}^{-4}\ \mathrm{m}^{-3}$.

- **Radiation pressure:**
  $P_{\mathrm{rad}} = U/3 = aT^4/3$.

Combining pressures

- **Total pressure from all components of a plasma consisting of a gas+radiation mix:**

$$P_{\mathrm{t}} = P_{\mathrm{gas}} + P_{\mathrm{rad}} = \underbrace{P_{\mathrm{i}}}_{\text{ions}} + \underbrace{P_{\mathrm{e}}}_{\text{electrons}} + P_{\mathrm{rad}}$$

- **Recall that internal temp of star $T_I \sim M/R$ and particle density $n \sim M/R^3$.**
- **Pressure ratio (classical gas/radiation):**

$$\frac{P_{\mathrm{rad}}}{P_{\mathrm{gas}}} = \frac{aT_{\mathrm{I}}^4/3}{n_{\mathrm{e}}kT_{\mathrm{I}} + n_{\mathrm{i}}kT_{\mathrm{I}}} \propto \frac{M^4/R^4}{M^2/R^4} \propto M^2.$$

- **Hence radiation pressure becomes increasingly important for increasing stellar mass, leading ultimately to instability at $M \sim 50\ M_{Sun}$.**

5.7 Density-temperature diagram