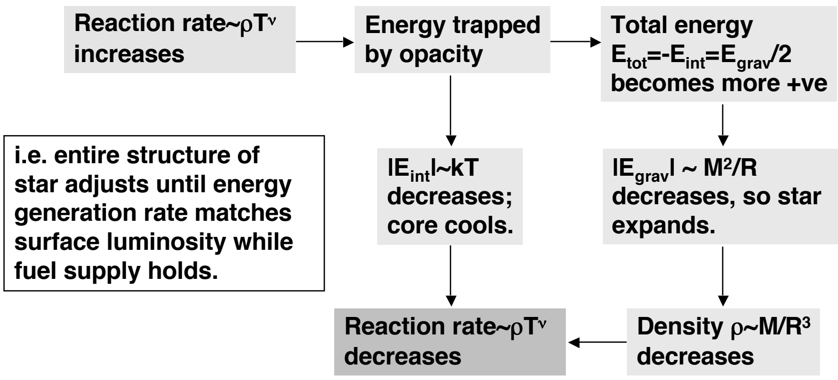
Main-sequence fusion control

- **Thermonuclear fusion acts as a thermostat.**
- **If non-degenerate core temperature rises:**



Simple models of stellar cores

- **To apply equation of state and nuclear rate equations we need to estimate central densities and temperatures.**
- **Need use this info to deduce:**
  - **Minimum and maximum mass of a star**
  - **How thermonuclear reactions regulate temperature**
  - **Why stars become red giants**
  - **Why the "helium flash" occurs**
  - **Chandrasekhar limit for WD masses**

Polytropic gas spheres

- **Rearrange hydrostatic equilibrium:**

$$m = -\frac{dP}{dr}\frac{r^2}{\rho G} \Rightarrow \frac{dm}{dr} = -\frac{d}{dr}\left(\frac{dP}{dr}\frac{r^2}{\rho G}\right)$$

- **Differentiate and use mass continuity:**

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi\rho G$$

- **Polytropic equation of state:**

$$P = K\rho^\gamma \Rightarrow \frac{dP}{dr} = \gamma K\rho^{\gamma-1}\frac{d\rho}{dr}$$

- **Eliminate P to get:**

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\gamma K\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi\rho G$$

Boundary conditions:
$\rho = \rho_c$ at r = 0
$d\rho/dr = 0$ at r = 0

Polytropes and Clayton models

- **Can reduce to a dimensionless form known as the Lane-Emden equation.**
- **Families of solutions for different γ are called polytropes. Messy for all but a few values of γ.**
- **Simpler approach devised by Clayton in 1986:**
  - **Parametrize pressure profile P(r) within star.**
  - **Use constraints imposed by hydrostatic equilibrium at surface and centre.**
- **Near centre:**

$$m(r) \approx \frac{4}{3}\pi r^3\rho_c \Rightarrow \frac{dP}{dr} = -\frac{Gm(r)\rho_c}{r^2} = -\frac{4}{3}\pi r G\rho_c^2$$

- **Near surface:**

$$\frac{dP}{dr} = -\frac{GM\rho(r)}{r^2}$$

A suitably smooth guess

- **Invent a function that mimics form of dP/dr, falling to zero at both centre and surface:**

$$\frac{dP}{dr} = -\frac{4\pi}{3} G\rho_c^2 r \exp\left[-\frac{r^2}{a^2}\right]$$

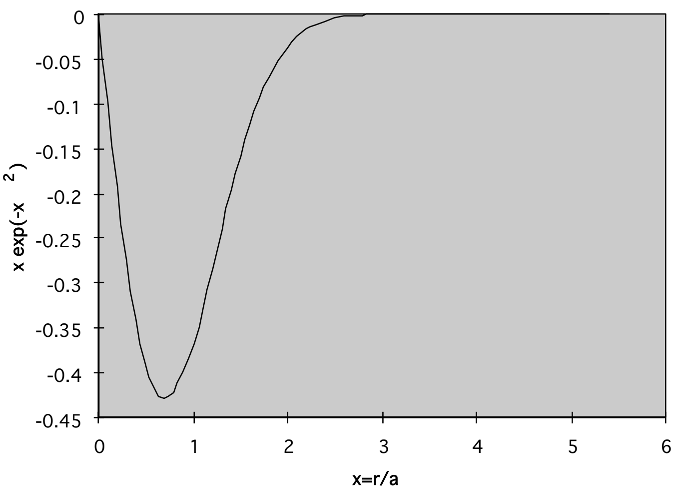
Length parameter, yet to be specified

- **Good representation at small r, *very* approximate at large r.**
- **Small gradient near surface is reproduced provided a << R.**
- **Integrate imposing P=0 at r=R as boundary condition:**

$$P(r) = \frac{2\pi}{3} G\rho_c^2 a^2 \left[\exp\left(-\frac{r^2}{a^2}\right) - \exp\left(-\frac{R^2}{a^2}\right)\right]$$

Pressure gradient

Plot of $\frac{dP}{dr} \Big/ \frac{4\pi}{3} G\rho_c^2 a = xe^{-x^2}$ over $0 < x < 5.4$.



Enclosed mass at radius r

- **Use hydrostatic equilibrium:**

$$Gm\,dm = -4\pi r^4 dP.\ \text{Integrate to get:}$$

$$\frac{Gm^2}{2} = -4\pi\int_0^r s^4 \frac{dP}{ds}ds.\ \text{Substitute for } \frac{dP}{ds}:$$

$$= 4\pi\frac{4\pi}{3}G\rho_c^2\int_0^r s^5 \exp\left[-\frac{s^2}{a^2}\right]ds$$

$$\text{Hence}\quad m = \frac{4\pi}{3}a^3\rho_c\Phi(x),$$

x=r/a

$$\text{where } \Phi^2(x) \equiv 6\int_0^x s^5 e^{-s^2}ds = 6 - 3(x^4 + 2x^2 + 2)e^{-x^2}.$$

Density and temperature

- **Get density directly from mass equation:**

$$\rho = \frac{1}{4\pi r^2}\frac{dm}{dr} = \frac{1}{4\pi a^3}\frac{1}{x^2}\frac{dm}{dx}$$

$$= \frac{\rho_c}{3x^2}\frac{d\Phi}{dx} = \frac{\rho_c}{3x^2}\left[\frac{1}{2\Phi}\frac{d}{dx}\left(\Phi^2\right)\right]$$

$$= \frac{\rho_c}{3x^2}\left[\frac{6x^5e^{-x^2}}{2\Phi}\right] = \rho_c\left[\frac{x^3e^{-x^2}}{\Phi}\right]$$

- **For ideal classical gas:**

$$T(r) = \frac{\overline{m}P(r)}{k\rho(r)} \text{ where } \overline{m} = \frac{2m_{\mathrm{H}}}{1+3X+0.5Y}$$

Approximate values in core

- **Clayton models are most reliable at small x=r/a where we can use series expansions* for Φ(x), ρ and T to get:**

$$\rho(r) = \rho_c\left(1 - \frac{5}{8}\frac{r^2}{a^2} + \frac{119}{640}\frac{r^4}{a^4} - ...\right) \text{ and, using}$$

$$P(r) \approx P_c \exp\left(-\frac{r^2}{a^2}\right) \text{ (OK provided a << R),}$$

$$T(r) = T_c\left(1 - \frac{3}{8}\frac{r^2}{a^2} + \frac{51}{640}\frac{r^4}{a^4} - ...\right)$$

*** preferably using a symbolic algebra package if you're as impatient as me!**

Centrally condensed stars

- **If mass is strongly concentrated toward centre, a<<R and we can ignore terms in exp(-a²/R²).**
- **Hence get total mass of star:**

$$M = m(R) = \frac{4\pi}{3}a^3\rho_c\Phi\left(\frac{R}{a}\right) \approx \frac{4\pi a^3\rho_c\sqrt{6}}{3}.$$

- **To get density and enclosed mass at r=a, note:**

$$\Phi^2(1) = 6 - 15/e \text{ and } \Phi^2(R/a) \approx 6, \text{ so at } r = a:$$

$$\Rightarrow \rho(a) = \rho_c\left[\frac{e^{-1}}{\sqrt{6 - 15/e}}\right] = 0.53\rho_c \text{ and}$$

$$\text{and } \frac{m(a)}{M} = \sqrt{\frac{\Phi^2(1)}{\Phi^2(R/a)}} = 0.28.$$

Solar centre

- **At r=0,** $P_c = \frac{2\pi}{3} G\rho_c^2 a^2 .$
- **Eliminate a using expression for total mass:**

$$P_c = \left(\frac{\pi}{36}\right)^{1/3} GM^{2/3} \rho_c^{\ 4/3} \text{ (noting } \left(\frac{\pi}{36}\right)^{1/3} \approx 0.44).$$

- **Valid for any centrally condensed star of homogeneous composition.**
- **Reality check: standard numerical solar models with $M=M_{Sun}$, $R=R_{Sun}$ give $\rho_c=9\times10^4$ kg m^{-3}, so expression for total mass gives $a=R_{Sun}/5.4$**
- **Then $P_c=1.9\times10^{16}$ Pa (cf. 1.65×10^{16} Pa, std model)**
- **$T_c=16\times10^6$ K for X=0.71, Y=0.27 (cf. 13.7×10^6 K, std model).**