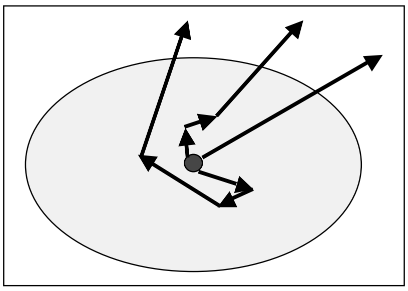
Monte Carlo Radiation Transfer I

- Monte Carlo “Photons” and interactions
- Sampling from probability distributions
- Optical depths, isotropic emission, scattering

Monte Carlo Basics

- Emit energy packet, hereafter a “photon”
- Photon travels some distance
- Something happens…



- Scattering, absorption, re-emission

Photon Packets

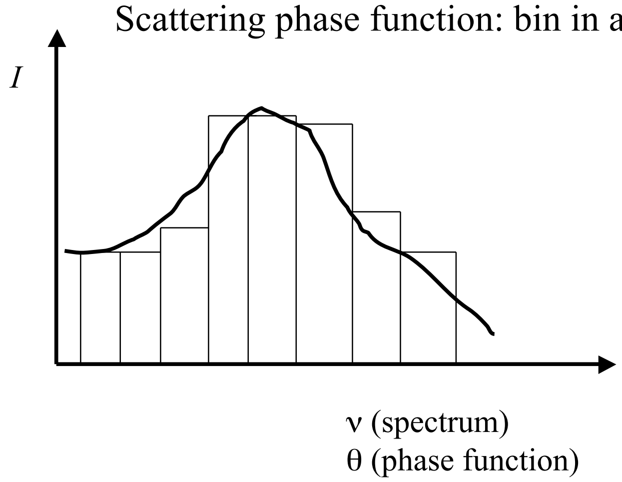
Total luminosity = L
each photon packet carries energy $E_i = L\ \Delta t / N$,
N = number of Monte Carlo photons.
MC photon represents N_γ real photons, where $N_\gamma = E_i / h\nu_i$
MC photon packet moving in direction θ contributes to the specific intensity:

$$I_\nu = \frac{\mathrm{d}E_\nu}{\cos\theta\ \mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\ \mathrm{d}\Omega}$$

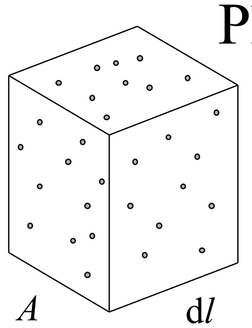
$$\Delta I_\nu = \frac{E_i}{\cos\theta\ \Delta A\ \Delta t\ \Delta\nu\ \Delta\Omega}$$

E_i → Energy packet

I_ν is a ***distribution function***. MC works with ***discrete*** energies. Binning the photon packets into directions, frequencies, etc, enables us to simulate a distribution function: Spectrum: bin in frequency
Scattering phase function: bin in angle



Photon Interactions



Volume = $A\,\mathrm{d}l$

Number density n

Cross section σ

Energy removed from beam per t / ν / $\mathrm{d}\Omega = I_\nu\,\sigma$

Number of photons absorbed/scattered from beam /sec

$$= I_\nu\,\sigma\,n\,A\,\mathrm{d}l$$

Number of photons absorbed/scattered from beam /sec/area

$$= I_\nu\,\sigma\,n\,\mathrm{d}l$$

Intensity differential over $\mathrm{d}l$ is $\mathrm{d}I_\nu = -\,I_\nu\,n\,\sigma\,\mathrm{d}l$. Therefore

$$I_\nu(l) = I_\nu(0)\exp(-n\,\sigma\,l)$$

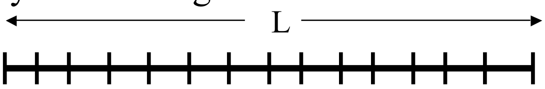
Fraction scattered or absorbed / length = $n\,\sigma$

$n\,\sigma$ = volume absorption coefficient = $\rho\,\kappa$

Mean free path = $1 / n\,\sigma$ = average dist between interactions

Probability of interaction over $\mathrm{d}l$ is $n\,\sigma\,\mathrm{d}l$

Probability of traveling $\mathrm{d}l$ without interaction is $1 - n\,\sigma\,\mathrm{d}l$



N segments of length L / N

Probability of traveling L before interacting is

$$P(L) = (1 - n\,\sigma\,L/N)(1 - n\,\sigma\,L/N)\ldots$$

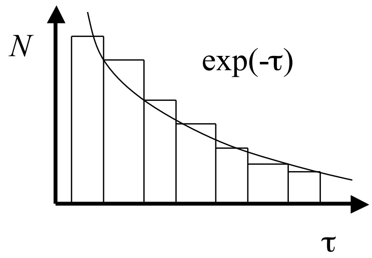
$$= (1 - n\,\sigma\,L/N)^N = \exp(-n\,\sigma\,L)$$

$$P(L) = \exp(-\tau)$$

τ = number of mean free paths over distance L.

Probability Distribution Function

PDF for photons to travel τ before an interaction is $\exp(-\tau)$. If we pick τ uniformly over the range 0 to infinity we will not reproduce $\exp(-\tau)$. Want to pick lots of small τ and fewer large τ. Same with a scattering phase function: want to get the correct number of photons scattered into different directions, forward and back scattering, etc.



Cumulative Distribution Function

$$\text{CDF} = \text{Area under PDF} = \int P(x)\,\mathrm{d}x$$

Randomly choose τ, θ, λ, … so that PDF is reproduced

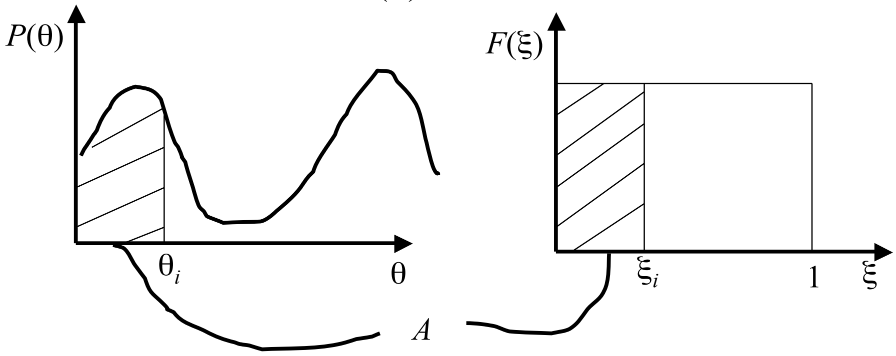
ξ is a random number uniformly chosen in range [0,1]

$$\xi = \int_a^X P(x)\,\mathrm{d}x \Rightarrow X$$

$$\int_a^b P(x)\,\mathrm{d}x = 1$$

This is the ***fundamental principle*** behind Monte Carlo techniques and is used to sample randomly from PDFs.

e.g., $P(\theta) = \cos\theta$ and we want to map ξ to θ. Choose random θs to "fill in" $P(\theta)$



$$\xi_i = \int_0^{\theta_i} P(\theta)\,\mathrm{d}\theta = \sin\theta_i \Rightarrow \theta_i = \sin^{-1}\xi_i$$

Sample many random θ_i in this way and "bin" them, we will reproduce the curve $P(\theta) = \cos\theta$.

Choosing a Random Optical Depth

$P(\tau) = \exp(-\tau)$, i.e., photon travels τ before interaction

$$\xi = \int_0^{\tau} \mathrm{e}^{-\tau}\,\mathrm{d}\tau = 1 - \mathrm{e}^{-\tau} \Rightarrow \tau = -\log(1-\xi)$$

Since ξ is in range [0,1], then $(1-\xi)$ is also in range [0,1], so we may write:

$$\tau = -\log\xi$$

Physical distance, L, that the photon has traveled from:

$$\tau = \int_0^{L} n\,\sigma\,\mathrm{d}s$$

Random Isotropic Direction

Solid angle is $d\Omega = \sin\theta \, d\theta \, d\phi$, choose (θ, ϕ) so they fill in PDFs for θ and ϕ. $P(\theta)$ normalized over $[0, \pi]$, $P(\phi)$ normalized over $[0, 2\pi]$:

$$P(\theta) = \tfrac{1}{2}\sin\theta \qquad P(\phi) = 1 / 2\pi$$

Using fundamental principle from above:

$$\xi = \int_0^\theta P(\theta)\,d\theta = \frac{1}{2}\int_0^\theta \sin\theta\,d\theta = \frac{1}{2}(\cos\theta - 1)$$

$$\xi = \int_0^\phi P(\phi)\,d\phi = \frac{1}{2\pi}\int_0^\phi d\phi = \frac{\phi}{2\pi}$$

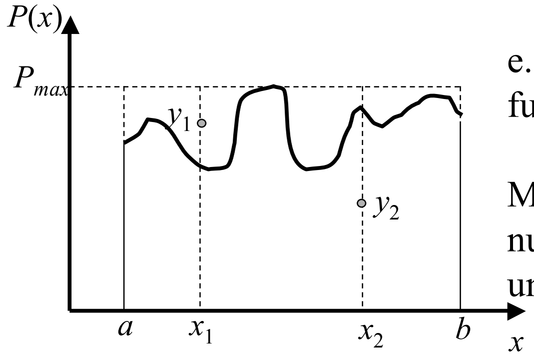
$$\theta = \cos^{-1}(2\xi - 1)$$

$$\phi = 2\pi\,\xi$$

Use this for emitting photons isotropically from a point source, or choosing isotropic scattering direction.

Rejection Method

Used when we cannot invert the PDF as in the above examples to obtain analytic formulae for random θ, λ, etc.



e.g., $P(x)$ can be complex function or tabulated

Multiply two random numbers: uniform probability / area

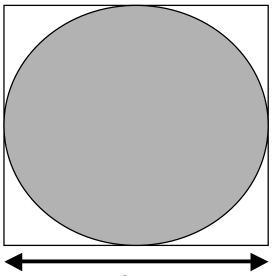
Pick x_1 in range $[a, b]$: $x_1 = a + \xi(b - a)$, calculate $P(x_1)$

Pick y_1 in range $[0, P_{max}]$: $y_1 = \xi \, P_{max}$

If $y_1 > P(x_1)$, reject x_1. Pick x_2, y_2 until $y_2 < P(x_2)$: accept x_2

Efficiency = Area under $P(x)$

Calculate π by the Rejection Method



Pick N random positions (x_i, y_i):
x_i in range $[-R, R]$: $x_i = (2\xi - 1) R$
y_i in range $[-R, R]$: $y_i = (2\xi - 1) R$
Reject (x_i, y_i) if $x_i^2 + y_i^2 > R^2$
Number accepted / $N = \pi R^2 / 4R^2$

$$N_A / N = \pi / 4$$

Increase accuracy (S/N): large N

FORTRAN 77:

```
do i = 1, N
   x = 2.*ran – 1.
   y = 2.*ran –1.
   if ( (x*x + y*y) .lt. 1. ) NA = NA + 1
end do
pi = 4.*NA / N
```

Albedo

Photon gets to interaction location at randomly chosen τ, then decide whether it is scattered or absorbed. Use the ***albedo*** or ***scattering probability***. Ratio of scattering to total opacity:

$$a = \frac{\sigma_S}{\sigma_S + \sigma_A}$$

To decide if a photon is scattered: pick a random number in range [0, 1] and scatter if $\xi < a$, otherwise photon absorbed

Now have the tools required to write a Monte Carlo radiation transfer program for isotropic scattering in a constant density slab or sphere