Monte Carlo RT

The traditional transfer equation is

$$\hat{n}\cdot\vec{\nabla} I_\nu = -\chi_\nu p I_\nu + j_\nu + \sum_i \eta_i \sigma_\nu^i \int \left(\frac{1}{\sigma_\nu^i}\frac{d\sigma_\nu^i}{d\Omega}\right) I_\nu(\hat{n}')\, d\Omega'$$

$q(\hat{n}',\hat{n})$ = "phase function"

absorption, emission, scattering

This is actually the same as the Boltzmann transport equation for particles; i.e., photons

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f = \left(\frac{Df}{Dt}\right)_{\text{interaction}}$$

f = phase space # density of photons

since the specific intensity

$$I_\nu(\vec{x},\hat{n}) = \frac{dE}{dA_\perp\, dt\, d\nu\, d\Omega}$$

$$= c\, h\nu\, f$$

photons travel with velocity

$$\vec{v} = c\hat{n}$$

and in steady-state $\frac{\partial f}{\partial t} \to 0$

So

Transport eqn = Boltzmann equation

Physically the transfer equation says

i) In absence of interactions the number of photons along any ray is constant; photons travel in straight lines
ii) Absorption removes photons from the beam = destruction
iii) emission adds photons to beam = creation
iv) scattering removes photons from one direction, changes them to another = scattering conserves photon #

Traditional Solution to RT equation:

- integrate along path (ray tracing)

$$I_\nu = e^{-\tau} I_\nu^\circ + \int S e^{-\tau'} d\tau'$$

- Problem: $S = \dfrac{j_\nu + \int n \sigma_{\nu,\theta} I_\nu}{\chi_\nu}$

depends on solution for I_ν from all other directions at each point along line of sight

∴ must integrate every direction, at every point for every frequency + iterate! (lambda-iteration)

- Problem: 1) 6-D integration is computationally expensive & memory intensive
  3) lambda-iteration fails without special techniques that enforce energy (flux) conservation
  2) Temperature + level populations depend on radiation field ⇒ opacity and emissivity not known apriori.

Monte Carlo RT

- Probabalistic simulation of the transport of photons (flow of energy) in the system
- It is an exact soln to the transfer eqn (using a statistical sampling of the paths)

Photon Packets

- Divide the total system luminosity L into N_γ packets of energy. The packet energy

$$E_\gamma = \frac{L \Delta t}{N_\gamma}$$

Δt = arbitrary simulation time

- Each packet is an ensemble of physical photons with the same position, direction, and frequency. The number of physical photons in the packet is

$$n = E_\gamma / h\nu$$

- Since each packet is an ensemble of particles it may be partially polarized. Normalizing the Stokes parameters to the packet energy

$$I = 1$$
$$Q = (E_{\leftrightarrow} - E_{\updownarrow})/E_\gamma$$
$$U = (E_{\nearrow} - E_{\nwarrow})/E_\gamma$$
$$V = (E_{\circlearrowright} - E_{\circlearrowleft})/E_\gamma$$

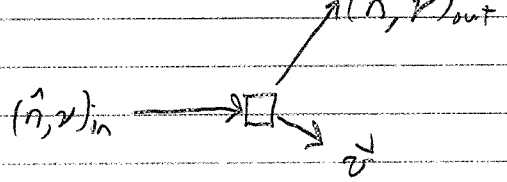
Photon Weights

- Sometimes we need to use photon packets with unequal energies
- Define the "weight" of the packet

$$w = \frac{\text{packet energy}}{E_\gamma}$$

- Example: Coherent scattering in moving media

In the rest frame of the gas, photon energy is conserved; in lab frame number of photons is unchanged



However photons are Doppler shifted, so new packet energy is

$$E_\gamma^{out} = \left(\frac{\nu_{out}}{\nu_{in}}\right) E_\gamma^{in}$$

so the packet weight is updated

$$w_{out} = \left(\frac{\nu_{out}}{\nu_{in}}\right) w_{in}$$

After a number of scatterings

$$W = \prod_{\text{scatterings}} \left(\nu_{out}/\nu_{in}\right)$$

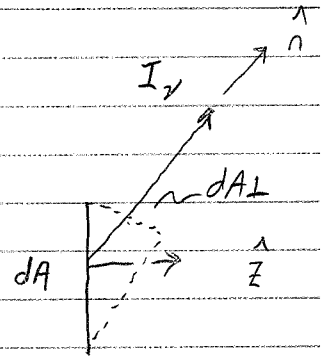
Note: 1) for non-relativistic motion this effect is often ignored
2) the total change in packet energy is the work done by the radiation force on the gas

MC Probabilities

- To simulate the transport of photons, we require the probabilities of photon emission and interaction
- For equal energy packets, number of packets is proportional to the energy

$\Rightarrow$ energy distributions = probability distributions

i) Emission from a planar surface



the specific intensity

$$I_\nu = \frac{dE/dt}{dA_\perp \, d\nu \, d\Omega}$$

but $dA_\perp = \mu \, dA$ $\quad \mu = \cos\theta = \hat{n}\cdot\hat{z}$

$$\therefore \mu I_\nu = \frac{dE/dt}{dA \, d\nu \, d\Omega}$$

So the probability distribution

$$\frac{dP}{dA \, d\nu \, d\Omega} \propto \mu I_\nu$$

integrating over all outward directions we find

$$\frac{dP}{dA \, d\nu} \propto \int_{2\pi} \mu I_\nu \, d\Omega = F_\nu \qquad (\text{or } H_\nu)$$

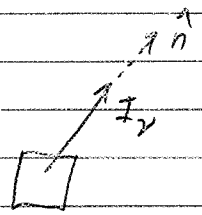
and then integrating over all frequencies

$$\frac{dP}{dA} \propto \int_0 F_\nu \, d\nu = F \qquad (\text{or } H)$$

ii) Emission from a volume element dV

The emissivity

$$j_\nu = \frac{dE/dt}{dV\,d\nu\,d\Omega}$$



so the probability distribution

$$\frac{dP}{dV\,d\nu\,d\Omega} \propto j_\nu$$

integrating over direction followed by frequency gives

$$\frac{dP}{dV\,d\nu} \propto 4\pi j_\nu$$

$$\frac{dP}{dV} \propto 4\pi \int_0^\infty j_\nu\, d\nu$$

iii) Star + Envelope

The total system luminosity

$$L = L_* + L_{env}$$

where $L_* = 4\pi \int H\, dA$

$L_{env} = 4\pi \int j_\nu\, d\nu\, dV$

The number of stellar and envelope photons is

$$N_* = (L_*/L)\, N_\gamma$$

$$N_{env} = (L_{env}/L)\, N_\gamma$$

so the probability of stellar and envelope photons is

$$P_* = L_*/(L_* + L_{env})$$

$$P_{env} = L_{env}/(L_* + L_{env}) = 1 - P_*$$

MCRT Algorithm

1) pick stellar vs envelope photon

ξ = rand (0,1) if $\xi < P_*$ then stellar photon
else envelope photon

2) pick starting position, frequency, and direction

a) Star

i) position, $\vec{x}_0$

$\frac{dP}{dA} = 4\pi H / L_*$ Note: H might be position-dependent (e.g., gravity darkening)

ii) Frequency, ν_0

$\frac{dP}{d\nu} = H_\nu / H$

iii) Direction, $\hat{n}$

$\frac{dP}{d\Omega} = \mu I_\nu / (4\pi H_\nu)$

b) Envelope

i) position

$\frac{dP}{dV} = 4\pi \int_0^\infty j_\nu d\nu / L_{env}$

ii) frequency

$\frac{dP}{d\nu} = j_\nu / \int_0^\infty j_\nu d\nu$

iii) direction

$\frac{dP}{d\Omega} = \frac{1}{4\pi}$

3) Doppler Shift photon frequency for source motion

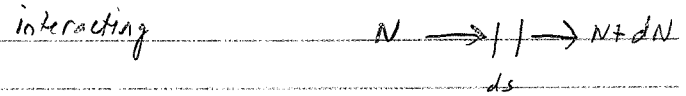
$$\nu \rightarrow (1 + \hat{n}\cdot\vec{v}/c)\nu_0$$

4) Transport photon along straight line to interaction location

Since $\frac{dI_\nu}{d\tau_\nu} = -\chi_\nu \rho I_\nu$, the probability for interaction is

$$dP = d\tau = \chi_\nu \rho ds$$

the number of photons interacting at a point is



$$dN = -N d\tau$$

where N = number passing that point

$$\Rightarrow N = N_0 e^{-\tau}$$

the cumulative probability of interaction is

$$P = \frac{N - N_0}{N_0} = 1 - e^{-\tau} \qquad \text{(Poisson distribution)}$$

generate uniform random number

$$\xi = \text{rand}(0,1) = 1 - e^{-\tau}$$

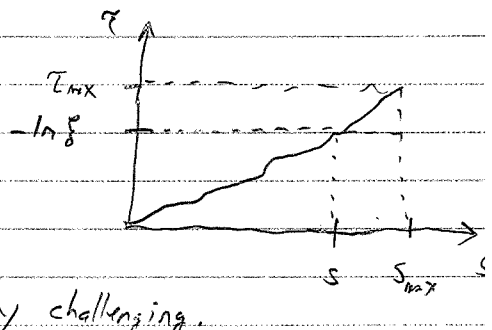
$$\tau = -\ln(1-\xi)$$

but $1-\xi$ is also uniformly distributed $(0,1)$

So equivalently

$$\tau = -\ln\xi$$

$$\int_0^s \chi_\nu \rho(\vec{x}_0 + s'\hat{n})ds' = -\ln\xi$$



Solve for s. This is computationally challenging. Analytic soln is faster than numerical

maximum value for s:

i) photon escape ⇒ place photon in observation bi

ii) photon hits star ⇒ reemit photon (locally) from stellar surface

otherwise, move photon to interaction location

$$\vec{x} = \vec{x}_0 + s\hat{n}$$

(Note: cartesian coords are usually used for this)

5) Find interaction process

The albedo is the scattering probability

$$a = \frac{\sigma_\nu}{\sigma_\nu + \kappa_\nu} = P_{scatter}$$

generate random number $\xi = rand(0,1)$

$\xi > a$ ⇒ absorb (i.e. destroy photon)

$\xi > a$ ⇒ scatter

6) Scatter photon

i) Doppler shift into rest frame of scatterer

ii) pick new direction

$$\frac{dP}{d\Omega} = \frac{1}{\sigma_\nu}\left(\frac{d\sigma_\nu}{d\Omega}\right) = g(\hat{n}_{in}, \hat{n}_{out})$$

iii) Doppler shift back to lab frame update photon weight

7) repeat $10^6 - 10^9$ times

Weighted (biased) Monte Carlo

- Suppose you want to send more photons to undersampled regions (e.g. interior of a dense cloud)

Answer: i) change scattering / emission probability distributions to send more photons in desired direction (i.e., bias the distribution functions).

ii) change photon weight by ratio of probabilities

$$w = \left(\frac{p_{normal}}{p_{biased}}\right) w$$

Making Measurements

- Sample Events:

  i) photon escape
  ii) photon interactions: absorption, scattering
  iii) cell/boundary crossing
  iv) photon motion

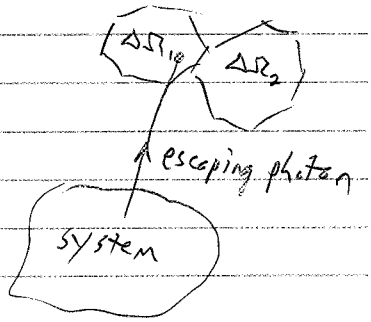
- Count samples

$$\text{energy} = \sum_i w_i E_\gamma$$

- Convert Energy / Counts to desired measurement

Examples

i) Emergent Flux



- divide emergent directions into solid angle bins, $\Delta\Omega_i$
- count # in each bin

$$N_i = \sum_{\substack{\text{escapes} \\ \text{into } i}} w$$

- For an observer at a distance d, the flux

$$F_i = \frac{dE}{dA\,dt} = \frac{N_i E_\gamma}{(d^2 \Delta\Omega_i)\Delta t} \qquad \text{but } E_\gamma = \frac{L\Delta t}{N_\gamma}$$

$$= \left(\frac{L}{d^2 \Delta\Omega_i}\right)\frac{N_i}{N_\gamma}$$

Normalizing to the stellar flux $F_* = \frac{L}{4\pi d^2}$

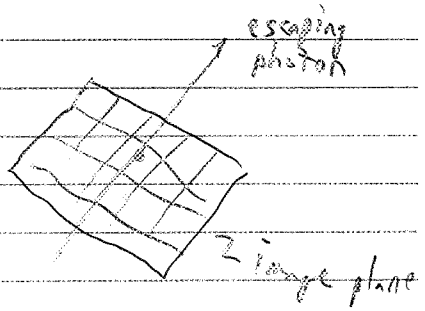
$$\left(\frac{F_i}{F}\right) = \left(\frac{4\pi}{\Delta\Omega_i}\right)\frac{N_i}{N_\gamma}$$

• To produce an SED / spectrum, bin in frequency and direction

$$\frac{F_\nu}{F_*} = \left(\frac{4\pi}{\Delta\Omega_i \Delta\nu_j}\right) \frac{N_{ij}}{N_\gamma}$$

• To produce an image, project escaping photon into pixel on an image plane.

Let k, l be the pixel indices.



The specific intensity

$$I_\nu = \frac{dE/dt}{\Delta A_i \Delta\nu_j \Delta\Omega_{kl}}$$

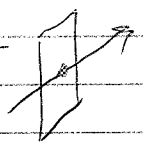
$$= \frac{N_{ijkl}(E_\gamma/\Delta t)}{d^2 \Delta\Omega_i \Delta\nu_j (A_{kl}/d^2)}$$

$$= \frac{L}{\Delta\Omega_i \Delta\nu_j A_{kl}} \left(\frac{N_{ijkl}}{N_\gamma}\right)$$

$$= \frac{L}{4\pi d^2} \left(\frac{4\pi d^2}{\Delta\Omega_i \Delta\nu_j A_{kl}}\right) \left(\frac{N_{ijkl}}{N_\gamma}\right)$$

$$\therefore \frac{I_\nu}{F_*} = \left(\frac{4\pi}{\Delta\Omega_i \Delta\nu_j \Delta\Omega_{kl}}\right) \frac{N_{ijkl}}{N_\gamma}$$

ii) Flux crossing a boundary



• construct a loop of area ΔA_i

• Count net number of photons crossing loop

$$N_i = \sum_{\substack{\text{outward} \\ \text{crossings}}} w - \sum_{\substack{\text{inward} \\ \text{crossings}}} w$$

• Flux normalized to stellar surface flux

$$\frac{F_i}{F_*} = \left(\frac{4\pi R^2}{L}\right) \frac{N_i(E_\gamma/\Delta t)}{A_i} = \left(\frac{4\pi R^2}{A_i}\right) \frac{N_i}{N_\gamma}$$

- Intensity moments

$$J/F_* = \frac{R^2}{A_i} \sum(w/\mu)$$

don't do this ← it has bad statistical properties

$$H/F_* = \frac{R^2}{A_i} \left(\sum w^+ - \sum w^-\right)$$

$$K/F_* = \frac{R^2}{A_i} \sum(\mu w)$$

Problems with Single Sampling

- The relative error for unweighted photons

$$S/N \sim \frac{1}{\sqrt{N_i}}$$

- oftentimes the number of samples is relatively small

$$N_i \sim \frac{N_{bins}}{N_\gamma}$$

for images $N_{bins} = N_{freq} \times N_{directions} \times N_{pixels}$

$$= 10^2 \cdot 10 \cdot 10^4 = 10^7$$

$\Rightarrow S/N = 100$ require 10^{11} photons.

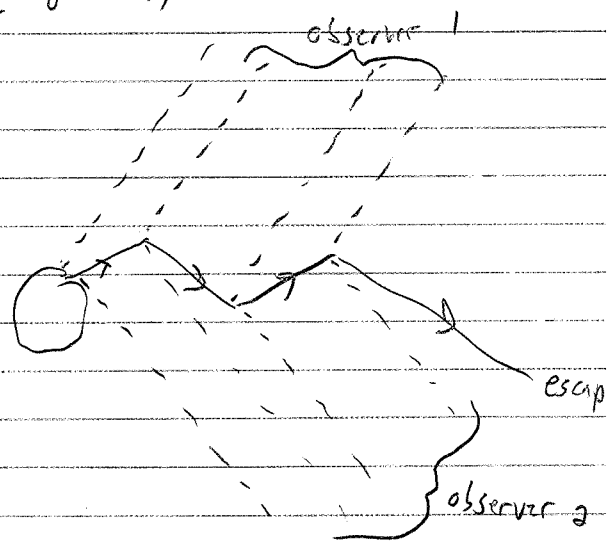
Advanced Sampling Techniques

- Source Function Sampling (also called peeling off)

i) sample photon interactions to measure incoming energy

ii) pick a set of observation directions

iii) calculate fraction of incoming energy escapes to each observer



a) photon emitted by star

$$N_i = \sum \frac{\mu I_\nu(\mu)}{4\pi H_\nu} e^{-\tau_{esc}} \cdot \Delta\Omega_i$$

b) photons emitted in envelope

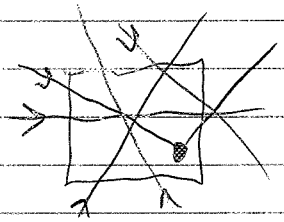
$$N_i = \sum \frac{W}{4\pi} e^{-\tau_{esc}} \Delta\Omega_i$$

c) photons scattered in envelope

$$N_i = \sum W g(\hat{n}_{in}, \hat{n}_{obs}) e^{-\tau_{esc}} \Delta\Omega_i$$

- Lucy Path Length Sampling

for optically thin cells very few photons interact but many photons pass through the cell



trick is to instead sample virtual absorptions using photon motion

$$N_{abs} = \sum d\tau$$

$$= \sum K_\nu \rho \, ds = K_\nu \rho \underbrace{\sum ds}$$

Error Estimation

A) Equal Energy (unweighted) Photons

- For a fixed number of photons, an escaping photon is either placed in a given bin of interest or it is not.
- The probability distribution is a binomial distribution
- The relative error

$$\frac{\delta E}{E} = \sqrt{(1 - N_i/N_\gamma)/N_i}$$

$$\approx 1/\sqrt{N_i} \qquad (N_i << N_\gamma)$$

B) Weighted Photons

- Each entire photon track is a statistically independent event
- Let

$$W_i = \sum_{\substack{\text{track} \\ \text{history}}} W_{obs} \qquad (E_{obs} = \sum_i W_i E_\gamma)$$

be the weight contributed by the i^{th} photon track

- The relative error is approximately

$$\frac{\delta E}{E} \approx \sqrt{\sum_i W_i^2} \Big/ \sum_i W_i$$

Problem: suppose one particular weight, W_i^{max}, is much larger than the rest

$$\Rightarrow \begin{cases} \sum_i W_i = W_i^{max} \\ \sum_i W_i = (W_i^{max})^2 \end{cases}$$

$$\therefore \frac{\delta E}{E} \to 1 \quad !!!$$

and the measurement is useless

Moral: when using weighted photons, make sure all weights contributing to a measurement all are roughly equal in magnitude.

Theorem: The minimum measurement error occurs when all photons have equal weights.

Unless you have no choice, always use unweighted photons

Radiative Equilibrium

- The circumstellar envelope is heated by the stellar radiation

  ⇒ Envelope emission is powered by stellar photons

  Note: there may be additional energy sources, e.g. gravitational potential energy in accretion

- Radiative Equilibrium

  energy absorbed = energy emitted

  This is both a global and local condition

- MC Radiative Equilibrium

  i) whenever a photon is absorbed reemit it locally

  ii) reduce L_{env} to include only energy from other sources (eg. accretion)

  ⇒ only photons generated in envelope are from other sources

  iii) choose frequency of reemitted photon from local emissivity

  $$\frac{dp}{d\nu} \propto j_\nu$$

  iv) outgoing direction is isotropic

  v) doppler shift as necessary

Note: 1) Energy is automatically and explicitly conserved

2) Since no energy is lost, lambda iteration converges (lower boundary necessarily communicates with outer boundary.)

• Radiative Equilibrium Temperature of Grid Cell

$E_{abs} = E_{emit}$

$n_{abs} E_\gamma / \Delta T = \int 4\pi j_\nu \, d\nu \, d\gamma$ but $j_\nu = \kappa_\nu \rho B_\nu(T_{cell})$

$= 4\pi \int \rho \, dV \int_0^\infty \kappa_\nu B_\nu(T_{cell}) \, d\nu$

$= 4\pi m_{cell} \kappa_P B(T_{cell})$ κ_P = Planck mean opacity

$= 4\pi m_{cell} \kappa_P \left(\frac{\sigma T_{cell}^4}{\pi}\right)$ note: κ_P is func of T_{cell}

$\sigma T_{cell}^4 = \frac{N_{abs}(E\nu/\Delta t)}{4 m_{cell} \kappa_P(T_{cell})}$

$\therefore \sigma T_{cell}^4 = \frac{L}{4 m_{cell} \kappa_P(T)} \left(\frac{N_{abs}}{N_\nu}\right)$

• How to measure N_{abs}

i) sample photon absorptions $N_{abs} = \Sigma W$

ii) use Lucy path length sampling

$N_{abs} = \Sigma W d\tau_{abs}$

$= \Sigma W \kappa_\nu \rho \, ds$

$= \rho \Sigma W \kappa_\nu ds$

$\sigma T_{cell}^4 = \frac{L \Sigma W \kappa_\nu ds}{4 V_{cell} \kappa_P(T_{cell}) N_\nu}$

Note: typically path length sampling has much smaller random measurement errors than counting absorptions, especially for optically thin cells

Temperature Correction

- In radiative equilibrium photon is emitted locally using j_ν to pick the frequency

Problem: $j_\nu = j_\nu(T_{cell})$
and we do not know T_{cell} apriori

Solution: 1) guess T and iterate

2) use instantaneous cell temperature and choose frequency to correct the frequency distribution (Bjorkman + Wood)

- Consider a single photon absorption event

$$T \rightarrow T + \Delta T$$

$$j_\nu \rightarrow j_\nu(T+\Delta T) = j_\nu(T) + \Delta T \frac{dj_\nu}{dT}$$

$$= \kappa_\nu B_\nu(T) + \Delta T \kappa_\nu \frac{dB_\nu}{dT}$$

The change in the frequency distribution is

$$\Delta j_\nu = \Delta T \kappa_\nu \frac{dB_\nu}{dT}$$

- Remit the photon choosing its frequency from

$$\frac{dP}{d\nu} = \kappa_\nu \frac{dB_\nu}{dT} \Big/ \int_0^\infty \kappa_\nu \frac{dB_\nu}{dT} d\nu$$

Note: 1) Iteration no longer required
2) This is an example of adaptive Monte Carlo
3) It works by reordering which frequencies are emitted first and by "filling in" the missing ones
4) it is not easily parallelized

Generally, Lucy path length sampling + iteration (with perhaps Bjorkman + Wood for the 1st iteration) is the method of choice.