Some relativistic metric MONDian extensions of gravity

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40 years of MOND
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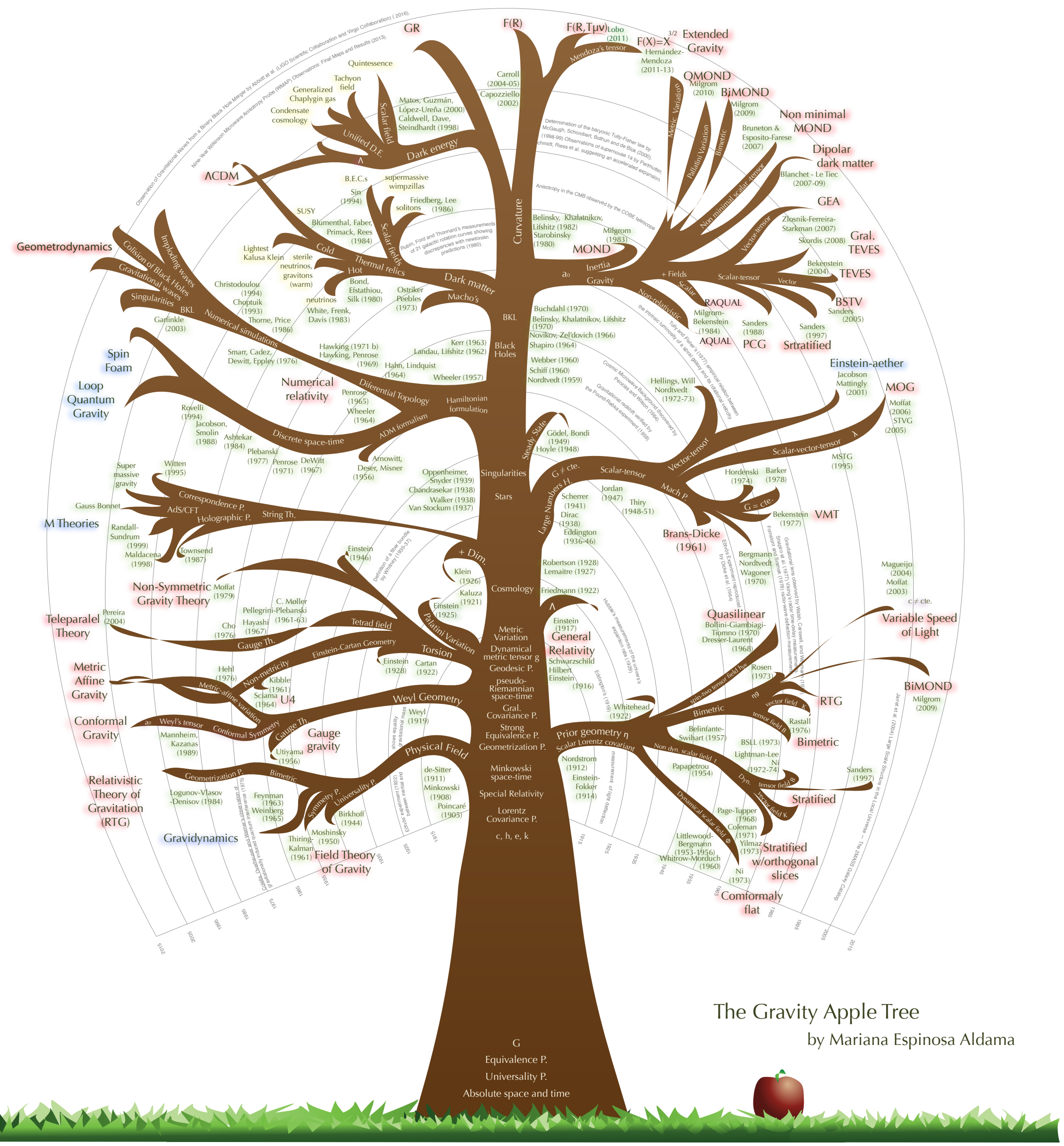
Gravity apple tree (Mariana Espinosa 2019)

https://prezi.com/rdkivznlhgga/the-gravity-apple-tree

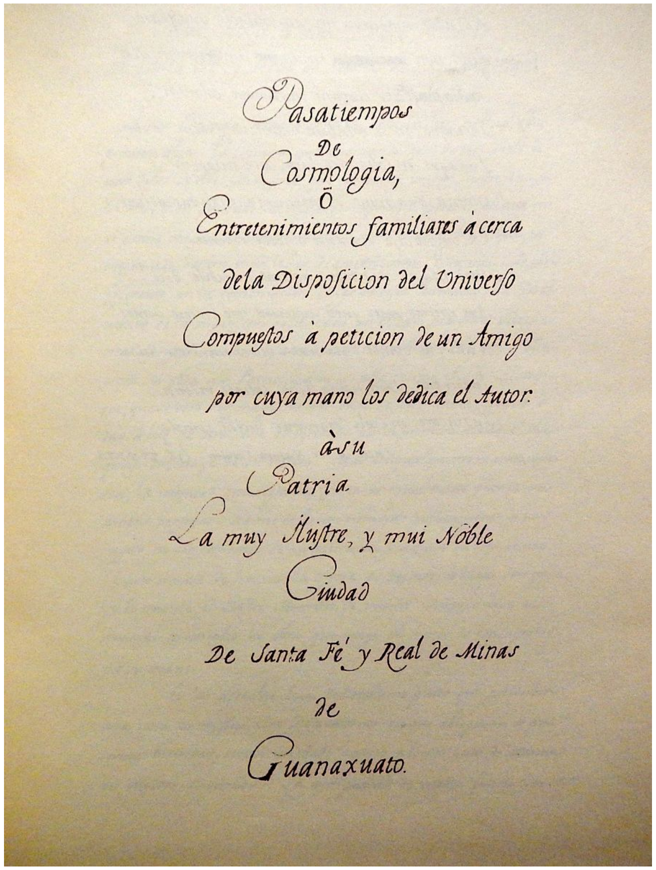


Casa abierta al tiempo
UNIVERSIDAD AUTÓNOMA METROPOLITANA

Genealogical tree for the theories of gravitation developed between 1905 and 2015. The time line is indicated by gray lines which have their base in two axis that point towards the theory of Special Relativity. Experiments and observations are located in the time-line space. Proponents for theories and dates for main published articles are colored in green. Principles followed by theories are written in the center of branches. Main theories are in red and models for dark matter and dark energy are in yellow. Approaches to quantum gravity are in blue. Special thanks to Sergio Mendoza, Mario Casanueva, Diego Méndez, Xavier Hernández and Shahem Hacyan for their advice and support.

https://prezi.com/rdkivznlhgga/the-gravity-apple-tree/

In México: Andrés Guevara y Basoasabal (1748-1801).


Pasatiempos
De
Cosmologia,
ó
Entretenimientos familiares à cerca
dela Disposicion del Universo
Compuestos à peticion de un Amigo
por cuya mano los dedica el Autor
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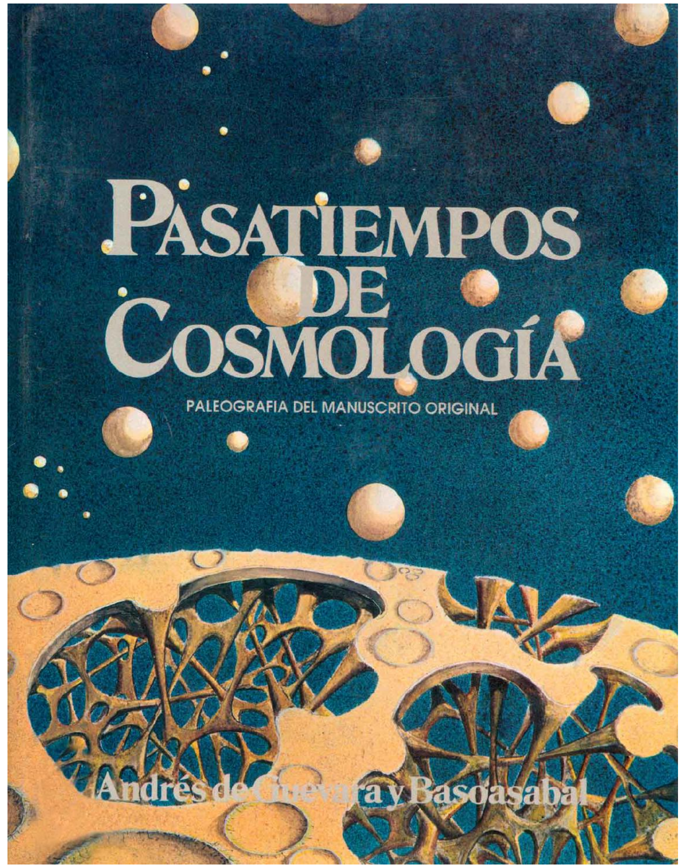
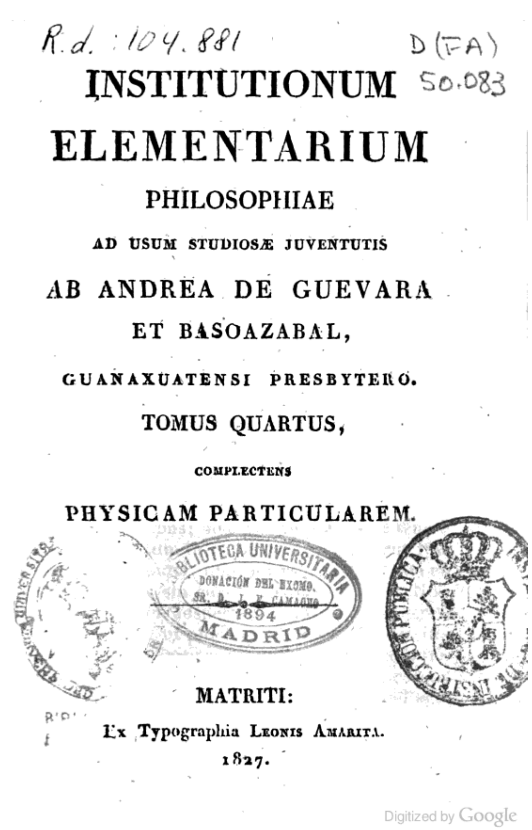


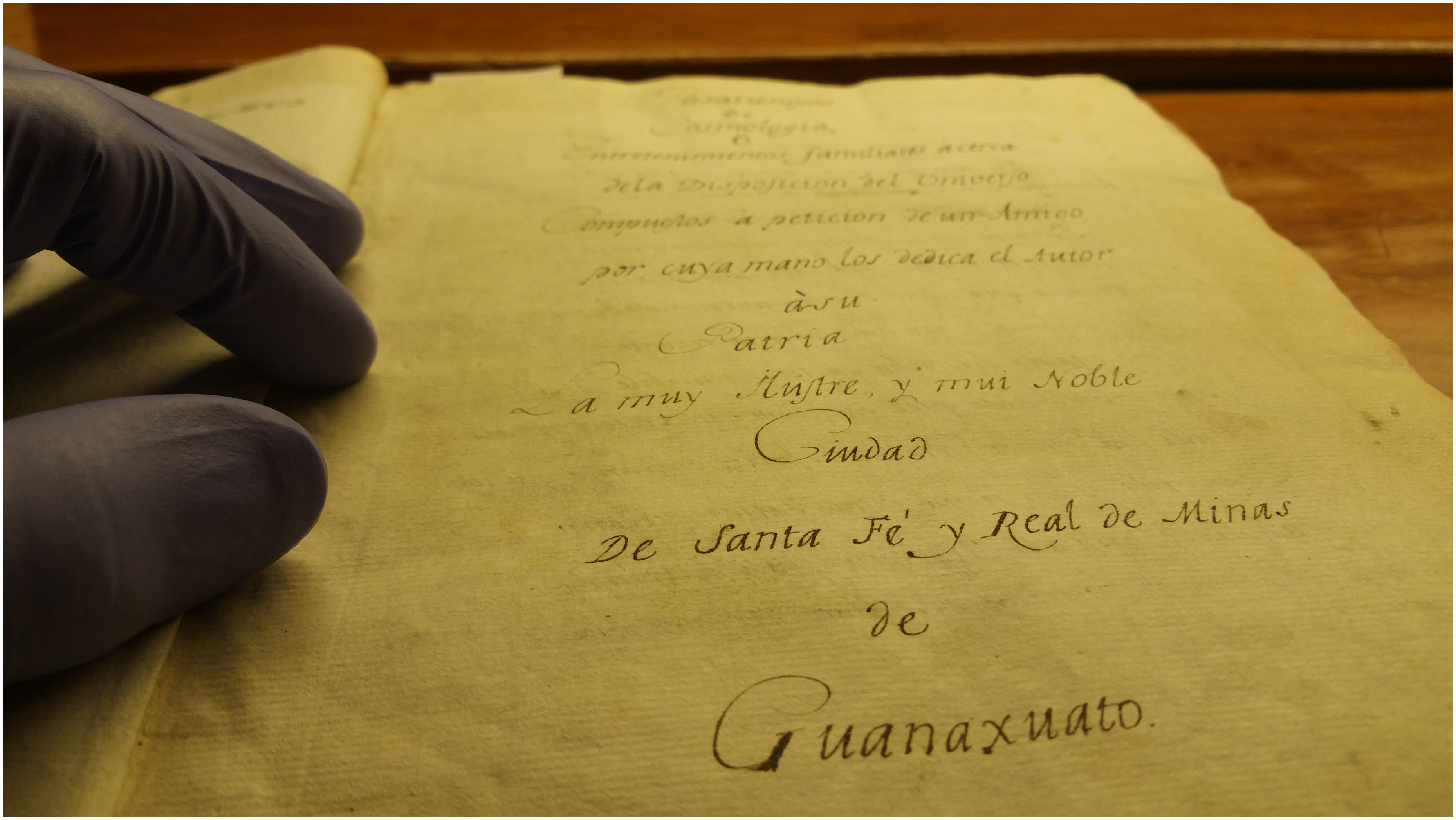


INSTITUTIONUM
ELEMENTARIUM
PHILOSOPHIAE
AD USUM STUDIOSÆ JUVENTUTIS
AB ANDREA DE GUEVARA
ET BASOAZABAL,
GUANAXUATENSI PRESBYTERO.
TOMUS QUARTUS,
COMPLECTENS
PHYSICAM PARTICULAREM.

MATRITI:
Ex Typographia Leonis Amaritae.
1827.

❄ Pasatiempos de cosmología o entretenimientos familiares acerca de la disposición del universo (1789)
Cosmological passtime or family entertainment about the disposition of the universe
❄ Instituonum Elementarium Philosophiæ(1796)

S. Galindo (2012) Revista Mexicana de Física, **58**-2.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Newtonian non-relativistic gravity (based on Kepler's 3rd law)	Extended non-relativistic gravity (based on Tully-Fisher's law

- Rotation curves (Kepler's third law):

$$v \propto \frac{M^{1/2}}{r^{1/2}}.$$

- Centrifugal balance $a \propto v^2/r$.
- Acceleration force is then:

$$a = -G_\mathrm{N} \frac{M}{r^2}.$$

- Calibrate with observations:

$$G_\mathrm{N} = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1}.$$

- Rotation curves (Tully-Fisher law):

$$v \propto M^{1/4}.$$

- Centrifugal balance $a \propto v^2/r$.
- Acceleration force is then:

$$a = -G_\mathrm{M} \frac{M^{1/2}}{r}.$$

- Calibrate with observations:

$$G_\mathrm{M} \approx 8.94 \times 10^{-11}\,\mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1/2}.$$

- Simplest form of MOND found since:

$$a_0 := \frac{G_\mathrm{M}^2}{G_\mathrm{N}}.$$

Milgrom (1983) obtained result requiring Newton's 2nd law to be modified. Since $(v^2/r) = a = GM/r^2$, then: $(v^2/r)^2 = a^2 = a_0\,GM/r^2$, i.e. MOdified Newtonian Dyn.

Relativistic Kepler's 3rd law

- Weakfield geodesic motion (massive particles):

$$g_{00} = 1 + \frac{2\phi}{c^2} = 1 - \frac{2G_{\mathrm{N}}M}{rc^2}.$$

- Isotropic coordinates:

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 - \left(1 - 2\gamma\phi/c^2\right)\delta_{kl}\mathrm{d}x^k\mathrm{d}x^l.$$

- Spherical coordinates:

$$g_{rr} = -1 - \frac{2\gamma G_{\mathrm{N}}M}{rc^2}.$$

Lensing observations imply $\gamma = 1$.
Schwarzschild solution of Einstein's field equations also imply $\gamma = 1$.

Relativistic Tully-Fisher law

- Weakfield geodesic motion (massive particles):

$$g_{00} = 1 + \frac{2\phi}{c^2} = 1 - \frac{2G_{\mathrm{M}}M^{1/2}}{c^2}\ln\left(\frac{r}{r_\star}\right).$$

- Isotropic coordinates:

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 - \left(1 - 2\gamma\phi/c^2\right)\delta_{kl}\mathrm{d}x^k\mathrm{d}x^l.$$

- Spherical coordinates:

$$g_{rr} = -1 - \frac{2\gamma G_{\mathrm{M}}M^{1/2}}{c^2}$$

Lensing observations imply $\gamma = 1$.
Mendoza et al. (2013)
Mendoza & Olmo (2015), Mendoza (2023)

- Lensing on elliptical, spiral and galaxy groups can be modelled using total matter distributions with isothermal profiles ($M_{\mathrm{T}} = v^2 r/G$) and DM profiles obey the same Tully-Fisher relation of baryonic matter of spirals: $v \propto M_{\mathrm{b}}^{1/4}$.
- Take GR -Schwarzschild- + DM.

$$g_{00\mathrm{S}} = 1/g_{11\mathrm{S}} = 1 - \frac{2r_{\mathrm{g}}}{r} = 1 - \frac{2GM_{\mathrm{T}}(r)}{c^2 r} = 1 - 2\left(\frac{v}{c}\right)^2 .$$

- The deflection angle $\beta_{GR} = F(g_{00\mathrm{S}}, g_{11\mathrm{S}}, r_i)$ can thus be calculated.
- This deflection angle is THE SAME for any metric theory of gravity and so $\beta_{GE} = \beta_{\mathrm{Ext}}$.
- Last relation is valid for all r_i and so, it is possible to find $g_{11\mathrm{Ext}}$ at $\mathcal{O}(2)$.

In short, Tully-Fisher law + lensing observations, at $\mathcal{O}(2)$ yield:

$$g_{00} = 1 - \frac{2G_{\mathrm{M}}M^{1/2}}{c^2} \ln\left(\frac{r}{r_\star}\right), \qquad g_{rr} = -1 - \frac{2G_{\mathrm{M}}M^{1/2}}{c^2}.$$

Hence: $\gamma = 1$ as in relativistic Kepler's 3rd law

Mendoza et al. (2013), Mendoza & Olmo (2015), Mendoza (2023)

Action and field equations

❄ General metric action with curvature-matter couplings (Harko & Lobo 2018):

$$S = \int F(R, \mathcal{L}_{\rm matt})\sqrt{-g}\,\mathrm{d}^4x, \qquad \text{where} \qquad T_{\mu\nu} := -\frac{2c}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\,\mathcal{L}_{\rm matt}\right)}{\delta g^{\mu\nu}}.$$

❄ Null variations of the action $\delta S/\delta g^{\alpha\beta} = 0$ yield field equations:

$$F_R R_{\alpha\beta} + \left(g_{\alpha\beta}\nabla^\mu\nabla_\mu - \nabla_\alpha\nabla_\beta\right)F_R - \frac{1}{2}\left(F - \mathcal{L}_{\rm matt}F_{\mathcal{L}_{\rm matt}}\right)g_{\alpha\beta} = \frac{1}{2}F_{\mathcal{L}_{\rm matt}}T_{\alpha\beta}$$

with a trace given by:

$$F_R R + 3\Delta F_R - 2\left(F - \mathcal{L}_{\rm matt}F_{\mathcal{L}_{\rm matt}}\right) = \frac{1}{2}F_{\mathcal{L}_{\rm matt}}T.$$

❄ The general non-geodesic motion of particles is:

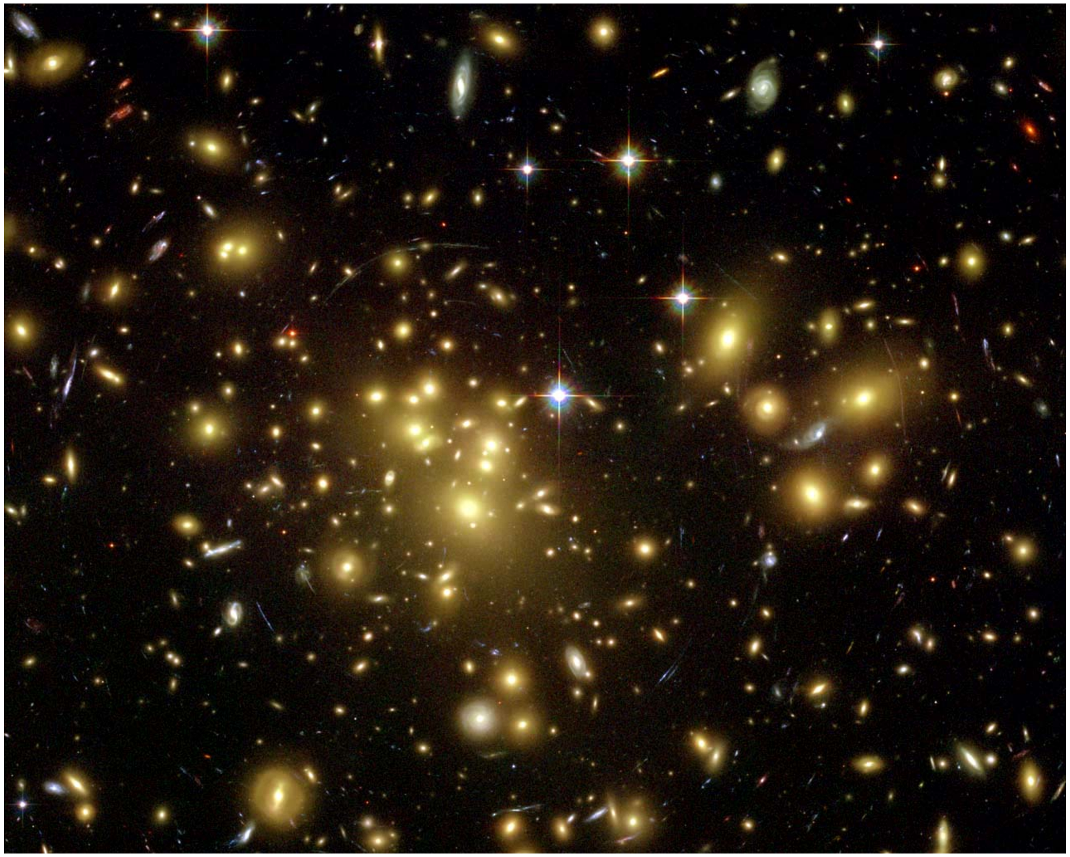
$$u^\alpha\nabla_\alpha u^\beta = \frac{\mathrm{D}u^\beta}{\mathrm{d}s} = \frac{\mathrm{d}^2x^\beta}{\mathrm{d}s^2} + \Gamma^\beta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \left(g^{\beta\lambda} - u^\beta u^\lambda\right)\nabla_\lambda\left(F_{\mathcal{L}_{\rm matt}}\frac{\mathrm{d}\mathcal{L}_{\rm matt}}{\mathrm{d}\rho}\right),$$

which follows from the fact that $\nabla_\alpha T^{\alpha\beta} \neq 0$.

Local Lagrangian: $\mathcal{L} \propto R^p \mathcal{L}^u_{\mathrm{matt}} + \mathcal{L}^v_{\mathrm{matt}}$ Deep MOND regime obtained when $p = -3$ and $v = u - 2$	
(i) Barrientos & Mendoza (2018) $\mathcal{L} \propto R^{-3} \mathcal{L}^3_{\mathrm{matt}} + \mathcal{L}_{\mathrm{matt}}$ (“**strong**” curvature-matter coupling)	$p = -3$ $u = 3$ $v = 1$
(ii) Barrientos, Bernal & Mendoza (2021) $\mathcal{L} \propto R^{-3} + \mathcal{L}^{-2}_{\mathrm{matt}}$ (“**weak**” curvature-matter coupling)	$p = -3$ $u = 0$ $v = -2$
Non-local Lagrangian: $\mathcal{L} \propto M^q R^p \mathcal{L}^u_{\mathrm{matt}} + \mathcal{L}_{\mathrm{matt}}$ Deep MOND regime obtained when $p = 6q - 3$ and $u = 3 - 4q$ (cf. Carranza & Mendoza (2013) $M(r) = 4\pi r^2 \int_0^r \rho(r) \mathrm{d}r$)	
(iii) Bernal, Capozziello, Hidalgo & Mendoza (2011) $\mathcal{L} \propto M^{3/4} R^{3/2} + \mathcal{L}_{\mathrm{matt}}$	$u = 0$ $p = 3/2$ $q = 3/4$

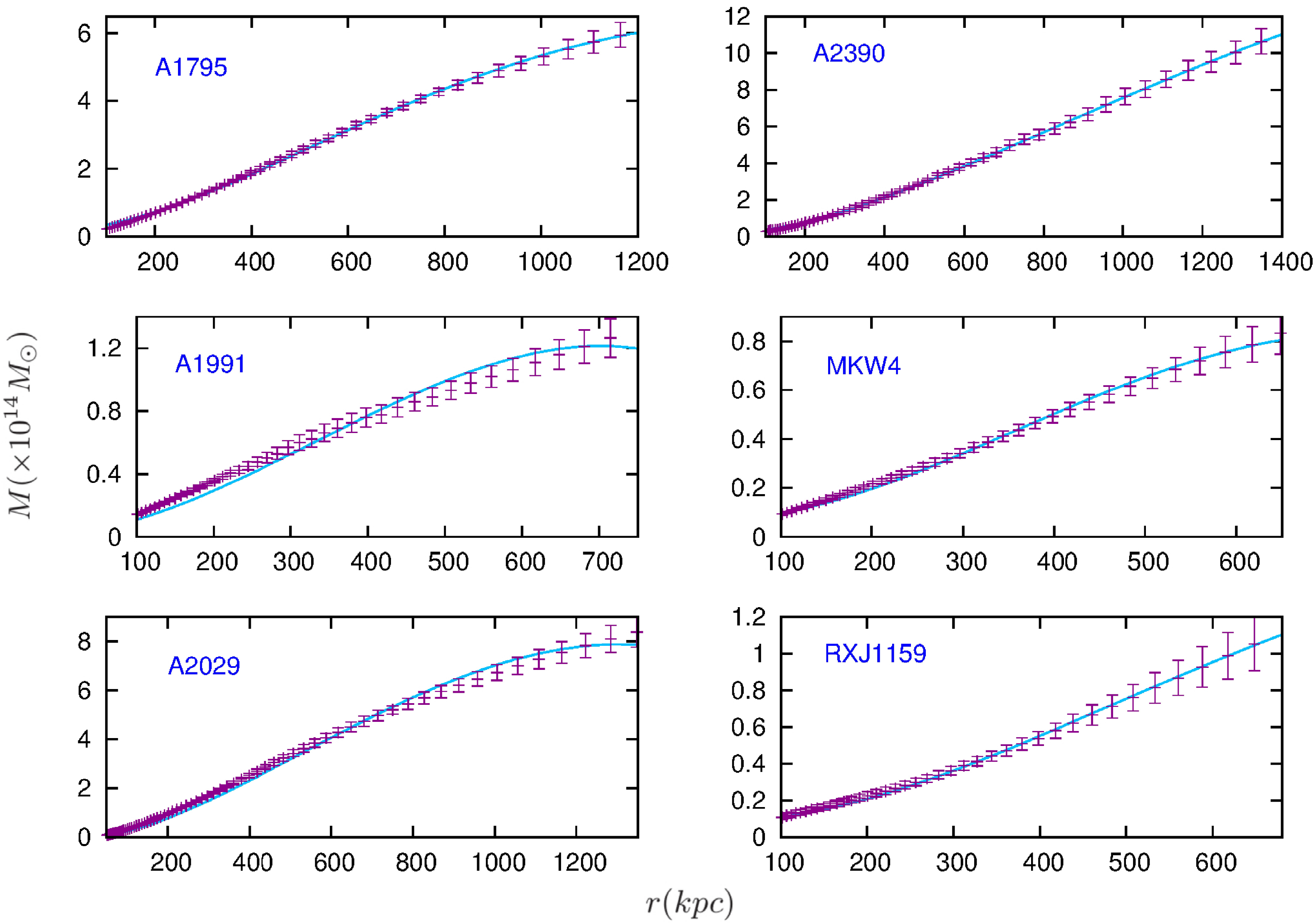
Clusters of galaxies (Bernal, Lopez-Corona & Mendoza 2019)

- Mercury's perihelium anomaly explained (mainly) by relativistic corrections since $v \sim 50\text{km/s} \Rightarrow v/c \sim 1.4 \times 10^{-4}$.
  For a cluster of galaxies $v \sim 1000\text{km/s} \Rightarrow v/c \sim 3.3 \times 10^{-3}$.



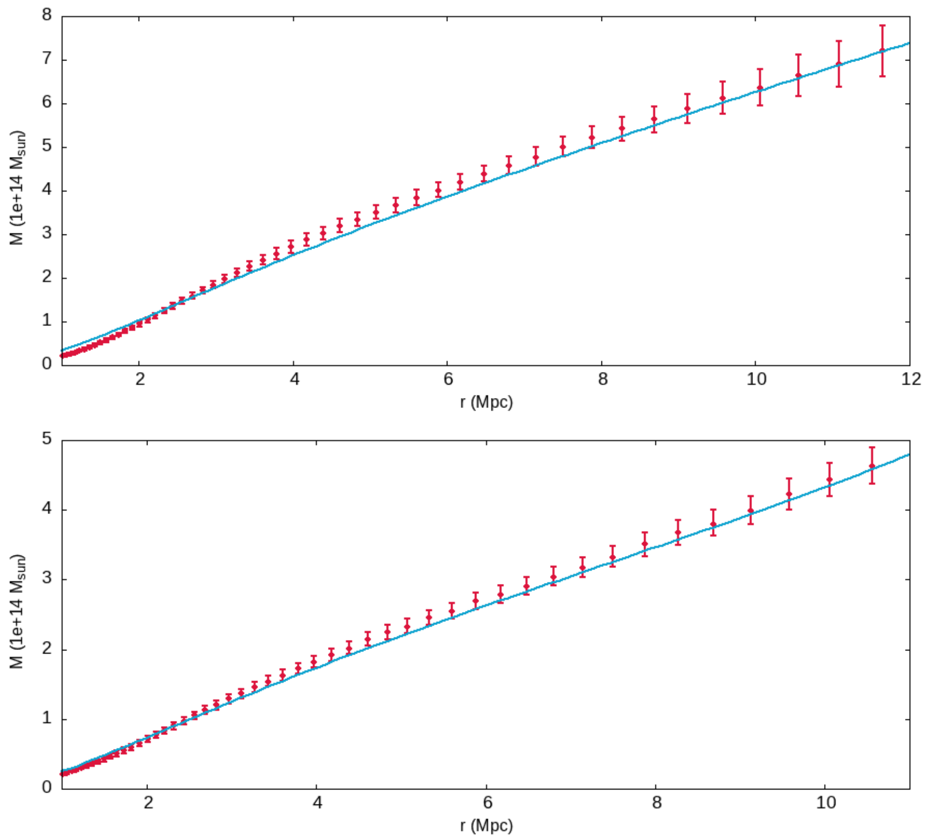
Acceleration from the geodesic equation at $O(4)$:

$$a = -\frac{c^2}{2}\left[{}^{(2)}g_{00,r} + \frac{r}{2}\,{}^{(2)}g_{00}\,{}^{(2)}g_{00,r} + {}^{(4)}g_{00,r}\right];$$



Using dimensional analysis in the deep MOND regime it is possible to construct a Post MONDian Parametrisation (PPM) in spherical symmetry at $\mathcal{O}(4)$ (Escoto & Mendoza 2023):

$$a = v^2/r = -\frac{\sqrt{GM(r)a_0}}{r} + (A^\star + B^\star \ln(r))\frac{GM(r)a_0}{c^2 r},$$



Cosmology

Present epoch Newtonian acceleration of the Universe with Hubble mass $M_{\rm H}$ is given by (cf. Bernal et al. 2011):

$$a \approx \frac{GM_{\rm H}}{R_{\rm H}^2} = \frac{G\left(c^3/GH_0\right)}{\left(c/H_0\right)^2} = cH_0 \approx 10^{-10}\text{m s}^{-2} \approx a_0.$$

Universe at the present epoch is in the deep MOND regime

$\Longrightarrow$ Simplest application: SNe Ia redshift – distance-modulus accelerated expansion.

$$\text{Cosmography: } a(t) = 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \ldots$$

$$H := \dot{a}/a, \quad q := -\ddot{a}H^{-2}/a, \quad j := \dddot{a}H^{-3}/a, \quad s := \ddddot{a}H^{-4}/a \ldots$$

Use FLRW metric for dust ($\mathcal{L}_{\text{matt}} = \rho c^2$ – Mendoza & Silva 2021) for the weak and strong curvature-matter field equations (turns out that mass conservation is valid).

Get a curvature-matter coupled "Friedmann" equation and use the standard cosmological results (cf. Peebles 1993):

- Distance modulus $\mu = m - M$:

$$\mu(z) = 5 \log \left[\frac{H_0 d_L(z)}{c} \right] - 5 \log h(z) + 42.3856.$$

- Luminosity-distance:

$$d_L(z) = \frac{c}{H_0} \left[z + \frac{1}{2}(1 - q_0) z^2 - \frac{1}{6} \left(1 - q_0 - 3q_0^2 + j_0 \right) z^3 + \frac{1}{24} \left(2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10q_0 j_0 + s_0 \right) z^4 + ... \right].$$

SNe Ia

Calibrate cosmographic parameters $H_0, q_0,\ j_0$ at present epoch using Union SNe Ia.

Results

$F \propto R^{-3} + \mathcal{L}_{\text{matt}}^{-2}$ (weak coupling)

$q_0 = -0.428081 \pm 0.05646$

$j_0 = -0.345827 \pm 0.08342$

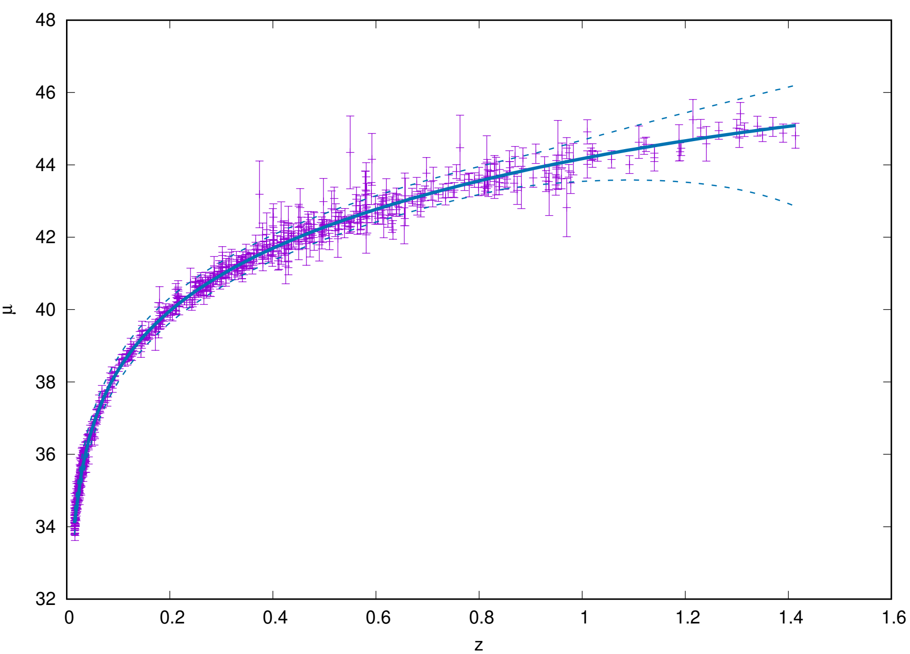
$H_0 = 69.511934^{+23.495497}_{-18.893123}$ km s^{-1}Mpc^{-1}

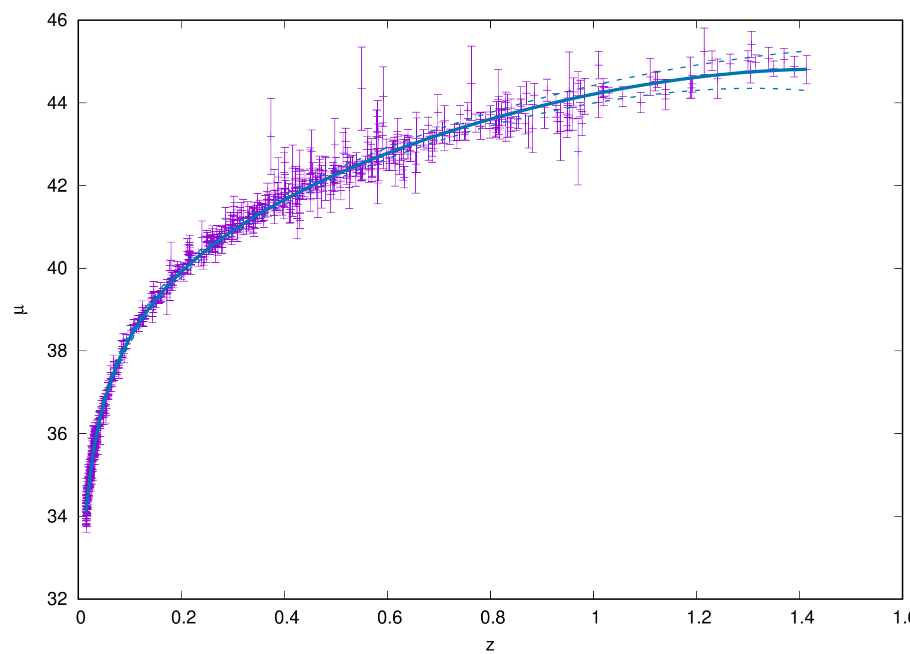
$F \propto R^{-3}\mathcal{L}_{\text{matt}}^{3} + \mathcal{L}_{\text{matt}}$ (strong coupling)

$q_0 = -0.417946 \pm 0.06891$

$j_0 = -0.240291 \pm 0.1493$

$H_0 = 70.363944^{+5.958497}_{-5.550341}$ km s^{-1}Mpc^{-1}





(Barrientos, Bernal & Mendoza 2021)

No dark matter, no dark energy!

Fractional Friedmann equations (non-local toy model)

(Barrientos, Mendoza, Padilla 2021)

Let $f(x) = x^k$ so that:

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} = \frac{k!}{(k-\alpha)!}x^{k-\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha}, \qquad I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau,$$

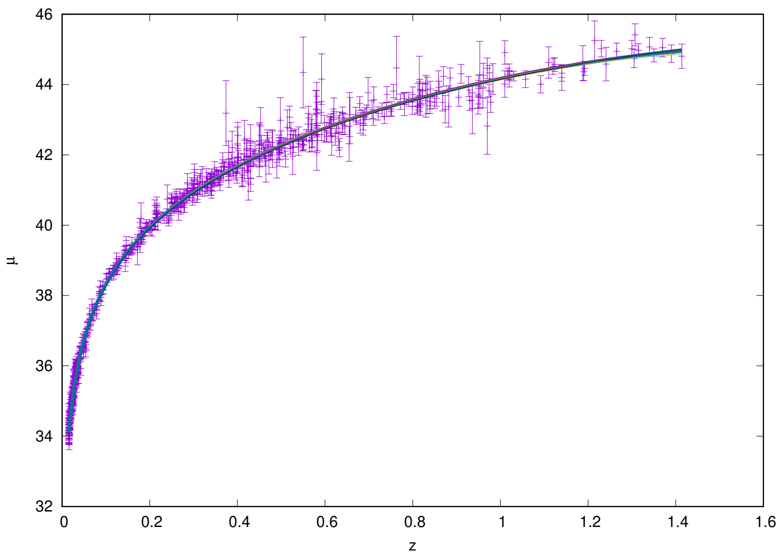
for any number α. **Last Step Modification:** change derivatives to unknown order γ and assume $a \propto t^n$.

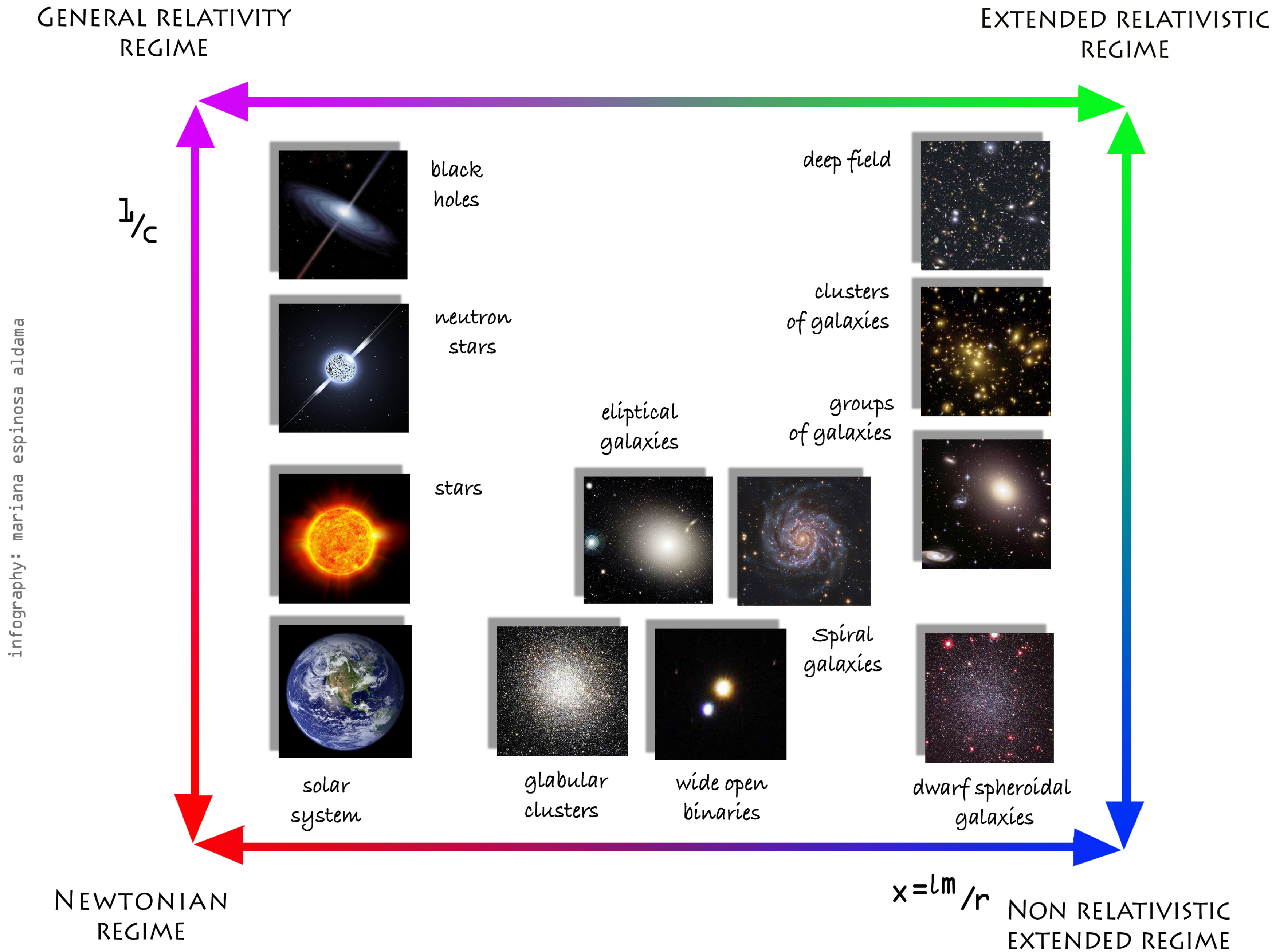
$n = 0.5539 \pm 0.0046$

$\gamma = 1.4937 \pm 0.0003 \approx 3/2$

$H_0 = 68.37 \mathrm{km/s/Mpc}.$

$\gamma = 3/2$ yields MOND (Giusti, 2020)!




General relativity regime
Extended relativistic regime
1/c
black holes
neutron stars
stars
solar system
deep field
clusters of galaxies
groups of galaxies
eliptical galaxies
spiral galaxies
glabular clusters
wide open binaries
dwarf spheroidal galaxies
Newtonian regime
x=lm/r
Non relativistic extended regime
infography: mariana espinosa aldama

Complete-AQUAL

Field equations for deep MOND regime should be of order $a^2/r \sim Ga_0\rho \sim Ga_0 M/r^3$ and so simplest field equation is:

$$\nabla \cdot (\phi|\nabla\phi|) \propto \rho,$$

which corresponds to the well known p-Poisson equation with p=3 that comes from the non-quadratic Lagrangian:

$$\mathcal{L}_{\mathrm{p}} = \kappa|\nabla\phi|^p + \phi\rho, \qquad \Longrightarrow \nabla \cdot \left(|\nabla\phi|^{(p-2)}\nabla\phi\right) = \frac{1}{p\kappa}\rho.$$

AQUAL is essentially a generalisation of the p-Poisson equation with:

$$\mathcal{L}_{\mathrm{AQUAL}} = F(|\nabla\phi|) + \phi\rho, \Longrightarrow \nabla \cdot \left(|\nabla\phi|^{-1} F'\, \nabla\phi\right) = \rho.$$

A complementary p-Poisson equation:

$$\nabla \cdot \left\{ \phi \nabla \left(|\nabla \phi|^{(p-2)} \right) \right\} = \frac{1}{p\kappa} \rho. \implies \ldots \mathcal{L}_{\text{p-complem}} = \kappa \nabla \left(|\nabla \phi|^{(p-2)} \right) \cdot \nabla \phi + \phi \rho,$$

The addition of the p-Laplacian and the complementary p-Laplacian yields the following complete p-Poisson field equation:

$$\nabla^2 \left(|\nabla \phi|^{p-2} \phi \right) = \frac{\rho}{\kappa p}, \quad \implies \quad \phi |\nabla \phi|^{(p-2)} = \frac{1}{2} |\nabla \phi^2| = \frac{4\pi}{p\kappa} \int \frac{\mathrm{d}V' \rho'}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

From now on, take $p = 3$ to get the deep MOND regime.

❄ Point mass source: $\rho = m\delta(\boldsymbol{r})$:

$$\phi |\nabla \phi| = |\nabla \phi^2|/2 \propto m/r, \implies \phi^2 \propto \ln(r), \quad \implies \quad a \propto m/r.$$

❄ Spherically symmetric configuration:

$$\phi \frac{\mathrm{d}\phi}{\mathrm{d}r} \propto \int \frac{\rho(\boldsymbol{r}'_{\text{int}})\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'_{\text{int}}|} - G \int \frac{\rho(\boldsymbol{r}'_{\text{ext}})\mathrm{d}V'}{|\boldsymbol{r} - \boldsymbol{r}'_{\text{ext}}|}.$$

❄ Binary system in the frame of reference of the centre of mass:

$$\phi \frac{\mathrm{d}\phi}{\mathrm{d}r} \propto \mu / r, \implies a \propto \mu^{1/2} / r.$$

with $\mu := m_1 m_2 / (m_1 + m_2)$ the reduced mass and $r := |\boldsymbol{r}_2 - \boldsymbol{r}_1|$ the separation between both masses. In other words, wide open binaries must show flat velocity profiles.

The complete p-Laplace equation can be generalised to a complete-AQUAL one:

$$\nabla^2 \left(|\nabla\phi|^{-\mathbf{1}} \, \mathbf{F}'(|\nabla\phi|) \, \phi \right) = \rho, \implies \mathcal{L}_{\text{c-AQUAL}} = F(|\nabla\phi|) + \nabla\phi \cdot \nabla F.$$

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