

# Following the mathematics of Hernquist 1990

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October 12, 2010

## 1 Purpose of this document

This guide is intended to be read alongside the paper Hernquist 1990 to help explain some of the rather lengthy mathematical results. It will not go into any depth on the physics content except where an understanding of the mathematics requires it. The only material used in addition to the paper itself is Galactic Dynamics (Binney and Tremaine).

This guide is meant to help you through some of more awkward and unfamiliar pieces of maths. To start with, though, most steps are explained as the basic concepts are dealt with. As the paper progresses I concentrate more on the new maths than describing, for a 5<sup>th</sup> time how to do integration by parts.

## 2 The model, its characteristics and the goal of the exercise

In order to understand the processes involved it is good to think for a moment about what the goal of the upcoming calculations are going to be. In this instance, we are interested in the use of this new model to find the distribution function of the spherical systems the Hernquist model describes.

The distribution function is a very important descriptor of the system. It represents coordinates in a six-dimensional position-velocity space (phase-space) such that at a given time  $t$  it can describe a sub-population of stars with position  $\mathbf{x}$  and velocity  $\mathbf{v}$ . It will become very important later. However, first we need to work out a few basic characteristics of the model before we can start making any progress towards the distribution function (DF).

The Hernquist model in terms of a density distribution is:

$$\rho(r) = \frac{M a}{2\pi r} \frac{1}{(r+a)^3} \quad (1)$$

where  $a$  represents a characteristic length scale of the system and  $M$  is the total mass of the system. The first step is to turn the density distribution into a cumulative mass distribution. This involves integrating the mass element

$$dm = \rho(r).dV \quad (2)$$

over all available volumes. This is a problem you will have almost certainly encountered a number of times before and it certainly doesn't look very challenging. I'm going to work through the problem in some detail to demonstrate that even apparently trivial problems, such as this, can throw up some unexpected problems in their solving. In this case, since the volume is spherical,

$$M(r) = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^r \frac{M a}{2\pi r} \frac{1}{(r+a)^3} \cdot r^2 \sin(\theta) dr.d\theta.d\phi = 2Ma \int_0^r \frac{r}{(r+a)^3} .dr \quad (3)$$

where the standard volume element in spherical coordinates has been applied. Integration of this proceeds easily enough using the by-parts method.

$$M \int_0^r \frac{2a.r}{(r+a)^3} .dr = [2a.r * -\frac{1}{2}(r+a)^{-2}]_0^r - M \int_0^r 2a. - \frac{1}{2}(r+a)^{-2} .dr \quad (4)$$

Tidying up and integrating the last term leaves us with

$$M(r) = [\frac{-ar}{(r+a)^2}]_0^r - [\frac{-a}{(r+a)}]_0^r \quad (5)$$

This leaves us with a slight problem that the equations produce infinities at  $r=0$ . The way to combat this is to recognise first that

$$\int_0^r = \int_0^\infty - \int_r^\infty \quad \text{and} \quad M_{tot} = \int_V \rho(r).dV = 4\pi \int_0^\infty \rho(r)r^2 .dr \quad (6)$$

This means we can re-write equation 6 in terms of the total mass and the new limits.

$$M(r) = \left[\frac{-ar}{(r+a)^2}\right]_0^r - \left[\frac{-a}{(r+a)}\right]_0^r = M_t - \left[\frac{-ar}{(r+a)^2}\right]_r^\infty - \left[\frac{-a}{(r+a)}\right]_r^\infty \quad (7)$$

So, now we don't have problems with infinities.

$$M(r) = M - M\left(\frac{ar + a(r+a)}{(r+a)^2}\right) = M\left(\frac{(r+a)^2 - r^2}{(r+a)^2}\right) = M\frac{r^2}{(r+a)^2} \quad (8)$$

The paper solves for the potential by integrating Poisson's equation for the system. A simpler, equivalent method is to use the standard classical result that  $F = \nabla\Phi$ . In this instance

$$F = \frac{-GM(r)}{r^2} \hat{\mathbf{r}} = \frac{\partial\Phi}{\partial r} \quad (9)$$

The answer here is clearly to integrate  $F$  with respect to  $r$ . However, the force-field created by a shell of potential at radius  $r$  affects space from  $r$  out to infinity so the integral wants to be between the limits of  $r$  and infinity. We have our expression for  $M(r)$  so we can express the potential thusly

$$\Phi = \int_r^\infty \frac{-GM}{(r+a)^2} = \left[\frac{GM}{(r+a)}\right]_r^\infty = \frac{-GM}{(r+a)} \quad (10)$$

The next few equations, producing the dimensionless variables are genuinely straightforward so we shall now move on to equation [9] on the paper and concern ourselves with the Jeans equation.

### 3 Velocity dispersion and anisotropy

For the next section we are interested in the velocity dispersion of the system. In order to study it we will be using something called a Jeans equation which is equation [9] in the paper. The derivation of a Jeans equation is long, detailed and not actually that relevant to the understanding of the paper. Because of this, the explanation of where the Jeans equations come from will be relegated to an Appendix. However, it is a good idea to look in more detail at what part velocity dispersion plays in the system as well as the new term,  $\beta$ .

The anisotropy parameter,  $\beta$ , describes the relative differences between the orthogonal components of motion in the system and it does this by examining the relations between the

velocity dispersions along different axes. However, there is one thing that needs examining in more detail at this moment and that is why the symbol,  $\bar{v}_x^2$ , is used to denote velocity dispersion rather than mean squared velocity. As it turns out the answer to this nit-picking question actually ties in to our original statement of the purpose and use of the DF.

Lets start by defining a useful function  $\nu(\mathbf{x})$  which describes the probability per unit volume of finding a star at  $\mathbf{x}$  but without caring what velocity it has i.e. a number density.

$$\nu(\mathbf{x}) = \int f(\mathbf{x}, \mathbf{v}).d^3\mathbf{v} \quad (11)$$

where  $f(\mathbf{x}, \mathbf{v})$  is our DF. The velocity dispersion is characterised by both the changes in the mean velocity from point to point and also the local spread of velocities around the mean at those points.

The mean velocity can be found by the familiar formula for expectation values

$$\bar{\mathbf{v}}(\mathbf{x}) = \int \mathbf{v} * P_x(\mathbf{v}).d^3\mathbf{v} \quad (12)$$

where  $P_x(\mathbf{v})$  is the probability distribution of velocities at the point  $\mathbf{x}$  and can be expressed as the probability of finding a star at  $\mathbf{x}$  with specific velocity  $\mathbf{v}$  over the probability of finding a star at  $\mathbf{x}$  with any velocity. We already have functions for both of those and so our mean velocity becomes

$$\bar{\mathbf{v}}(\mathbf{x}) = \frac{1}{\nu(\mathbf{x})} \int \mathbf{v} * f(\mathbf{x}, \mathbf{v}).d^3\mathbf{v} \quad (13)$$

Our velocity dispersion measure now has a description of the changes in mean velocity from point to point so now we just modify it so that the spread of velocities (the variance of  $\mathbf{v}$ ) is being used rather than specific values. This leads us to the velocity dispersion tensor

$$\sigma^2 \equiv \frac{1}{\nu(\mathbf{x})} \int (v - \bar{v})^2 . f(\mathbf{x}, \mathbf{v}).d^3\mathbf{v} \quad (14)$$

Notice that this essentially governed entirely by the form of the DF. Now we have a velocity dispersion measure and some idea of why being able to describe it is important in our attempt to find the DF there just remains the question of the confusion with the m.s velocity.

This is due to two things, the fact that the system is isotropic and governed by a single potential which is only a function of position and the fact that the description in words we

gave for the velocity dispersion can be written directly as

$$\sigma^2 = \overline{v^2} - \bar{v}^2 \quad (15)$$

The fact that the system is governed by a single function  $\Phi(\mathbf{x})$ , the Hamiltonian =  $K + V$  describes the motion. Replacing  $f(\mathbf{x}, \mathbf{v})$  with  $\frac{v^2}{2} + \Phi$  in equation 13 will describe the mean velocity. The integral is over all velocities but since  $\mathbf{v}$  is odd while our new  $f(H)$  is even, the overall integral is odd and will evaluate to 0. This means that  $\bar{v} \cdot \bar{v} = 0$  so from equation 15,  $\sigma^2 = \overline{v^2}$ . In other words, in this scenario the velocity dispersion and the mean squared velocity are equivalent. This finally brings us to understanding the anisotropy parameter:

$$\beta \equiv 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\overline{v_\theta^2} + \overline{v_\phi^2}}{2\overline{v_r^2}} \quad (16)$$

If orbits are perfectly circular then  $\overline{v_r^2} = 0$  and  $\beta = -\infty$ . If the orbits are radial then  $\overline{v_\theta^2} = \overline{v_\phi^2} = 0$  and  $\beta = 1$ . However, the case we are examining with the Hernquist is for orbits which are ergodic i.e. all the mean square velocity components are equal and  $\beta = 0$ .

There is one last discrepancy to clear up. We have so far been using  $\nu(\mathbf{x})$ . In actual fact this is equivalent to  $\rho r$  that you find in Hernquist 1990. The reason is that Hernquist states that he chose to normalise his density function to the total mass, not unity.

## 4 Velocity dispersion

Now we have a good idea of what's going on with anisotropy and the concerns of the previous section we can start making progress with the mathematics again. We were last considering the Jeans equation. Now, that we have an understanding of  $\beta$  we can, ironically, set it to 0 and remove middle term to obtain the equation we will be integrating

$$\frac{1}{\rho} \frac{d}{dr} (\rho \overline{v_r^2}) = -\frac{d\Phi}{dr} \text{ therefore } \overline{v_r^2} = \frac{1}{\rho} \int_r^\infty \rho(r) \frac{d\Phi}{dr} \quad (17)$$

where the integral is taken over those limits for similar reasons to the earlier evaluation of the potential.

The problem with this integral, and many of those to come is not that the integrals are particularly hard, it is just that they are very unwieldy. This will, in fact, be the last one solved 'by hand' as equation solvers such as Maple will be used for subsequent integrals. Since

a lot of the following proof is arithmetic I will skip over some sections to save everyone's time. So, using equations 1 and 10 we can begin

$$\bar{v}_r^2 = -\frac{GM^2}{2\pi\rho(r)} \int_r^\infty \frac{a}{r} \frac{1}{(r+a)^5} \cdot dr \quad (18)$$

This is best solved by partial fractions as the power of 5 makes it hard to find substitutions that improve matters. So, expanding the integrand gives

$$\frac{a}{r \cdot (r+a)^5} = \frac{A}{r} + \frac{B}{(r+a)} + \frac{C}{(r+a)^2} + \dots + \frac{F}{(r+a)^5} \quad (19)$$

Find the common factors, solve the 6 simultaneous equations (which are mercifully simple) and the result should be

$$\frac{a}{r \cdot (r+a)^5} = \frac{a^{-4}}{r} - \frac{a^{-4}}{(r+a)} - \frac{a^{-3}}{(r+a)^2} - \dots - \frac{1}{(r+a)^5} \quad (20)$$

This can be integrated, a term at a time, to yield another series which can be substituted into equation 18 to give us the final result

$$\bar{v}_r^2 = \frac{-GM^2}{2\pi\rho(r)} \left\{ a^{-4} \cdot \ln(r+a) - a^{-4} \cdot \ln(r) - \left[ \frac{1}{a^3(r+a)} + \frac{1}{2a^2(r+a)^2} + \frac{1}{3a(r+a)^3} + \frac{1}{4(r+a)^4} \right] \right\} \quad (21)$$

Expanding out the  $\frac{1}{\rho(r)}$  in the front of the equation leads to

$$\bar{v}_r^2 = \frac{-GM^2}{12a} \left\{ \frac{12r(r+a)^3}{a^4} \cdot \ln\left(\frac{r+a}{r}\right) - \frac{r(r+a)^3}{(r+a)} \left[ \frac{12}{a^3} + \frac{6}{a^2(r+a)} + \frac{4}{a(r+a)^2} + \frac{3}{(r+a)^3} \right] \right\} \quad (22)$$

The last stretch is to take the square bracketed term and change it into the form required by the result in Hernquist. This involves writing it as a single fraction over a common denominator of  $a^3(r+a)^3$ . Subsequent expansion of the numerator into a polynomial in  $a$  and  $r$  lets you remove the factor of  $a^3$  from the denominator to leave

$$\frac{12\left(\frac{r}{a}\right)^3 + 42\left(\frac{r}{a}\right)^2 + 52\left(\frac{r}{a}\right) + 25}{(r+a)^3} \quad (23)$$

which we can substitute into equation 22 to give the final result

$$\bar{v}_r^2 = \frac{-GM^2}{12a} \left\{ \frac{12r(r+a)^3}{a^4} \cdot \ln\left(\frac{r+a}{r}\right) - \frac{r}{(r+a)} \left[ 12\left(\frac{r}{a}\right)^3 + 42\left(\frac{r}{a}\right)^2 + 52\left(\frac{r}{a}\right) + 25 \right] \right\} \quad (24)$$

The last thing to do is to apply the limits of integration of *infinity*  $\rightarrow$   $r$ . Evaluating at infinity gives us 0 which effectively reverses the sign on the function subtracting it from 0.

## 5 Energies

From here on out the functions become much too tiresome to do by hand. For example, look at the definition of  $T(r)$  given in the paper. To evaluate even the first term,  $A.r^2 \cdot \ln(r+a)$  requires multiple instances of integration by parts. As such, Maple's interactive function solver will be used which should speed things up considerably.

The definition of  $T(r)$  seen here follows from simple enough logic.

$$T(v) = \frac{1}{2}m.v^2 \rightarrow T(r) = \int_0^r \frac{1}{2}M(r)v^2 \quad (25)$$

The only trick is to notice that, since our system is ergodic, all components of the velocity dispersion are equal and so

$$v^2 = \bar{v}_r^2 + \bar{v}_\theta^2 + \bar{v}_\phi^2 = 3\bar{v}_r^2 \quad (26)$$

Then the formula can be expressed in the usual spherical polars to produce

$$T(r) = 6\pi \int_0^r \rho \bar{v}_r^2 r^2 . dr \quad (27)$$

Substituting in for  $\rho$  and  $\bar{v}_r^2$  gives us the function we need to solve. To find a solution using Maple what we first do is define the functions Maple will be working with:

```
> rho := M*a/((2*pi)*r*(r+a)^3)
```

and so on for the other given functions. Finding

> Tintegrand := 6\*pi\*rho\*vsquared\*r^2

will produce a large, ungainly function that Maple will be unable to integrate. The easiest short-term solution is to break up the function into smaller pieces and let Maple integrate each part separately before adding them all up at the end. The downside to this approach is how tedious and inflexible it is: any change in an earlier part of the function will need to be copied manually down every time it appears. The final step is to try and express the result of this process into something that resembles the desired functional form.

The result in Maple for the above is a polynomial with 12 very similar terms in it. By way of example, the simplification of the terms in  $\ln(f(r))$  proceeded as follows

$$\left[-\frac{GM^2 \ln\left(\frac{a}{r}\right)}{a} + \frac{GM^2 r^3 \ln\left(1 + \frac{a}{r}\right)}{a^4} + \frac{GM^2 \ln\left(1 + \frac{a}{r}\right)}{a} - \frac{GM^2 \ln(r+a)}{a}\right]_0^r \quad (28)$$

$$= \frac{GM^2}{a} \left[\left(\frac{r}{a}\right)^3 \ln\left(1 + \frac{a}{r}\right) - \ln\left(1 + \frac{a}{r}\right) - \ln(a) + \ln\left(1 + \frac{a}{r}\right)\right]_0^r \quad (29)$$

The constant term will, by definition, be removed when the limits are applied and thus this section of the result is the same as that found by Hernquist. The same applies for the other terms.

Potential energy is much easier and can be solved without Maple. The total potential energy is given by the usual integration in spherical coordinates

$$\Omega_{tot} = \frac{1}{2} \int_V \rho(r)\phi(r).dV = 2\pi \int_0^\infty \rho(r)\phi(r).r^2 dr = -GM^2 a \int_0^\infty \frac{r}{(r+a)^4}.dr \quad (30)$$

where the prefactor of  $\frac{1}{2}$  is to prevent counting of the same particles (or, more exactly, the same mass-containing volume element) twice. Integration by parts will remove the factor of 'r' and, after a little work

$$\left[\frac{-r}{3(r+a)^3}\right]_0^\infty - \left[\frac{-1}{6(r+a)^2}\right]_0^\infty = [0 - 0] - \left[0 - \frac{-1}{6a^2}\right] = \frac{-GM^2}{6a} \quad (31)$$

The final piece of this section is to find the escape and circular velocities. The equations are given in the but it's worth a second or two to remind ourselves of where they come from.

The circular velocity comes from equating the centripetal force with the gravitational attraction thus



$$\frac{mv_c^2}{r} = \frac{GM(r)m}{r^2} \rightarrow v_c = \sqrt{\frac{GM(r)}{r}} \quad (32)$$

and the escape velocity is the velocity required to have exchanged all kinetic energy for potential at  $r = \infty$

$$\frac{1}{2}mv_e^2 = -m\phi(r) \rightarrow v_e = \sqrt{-2\phi(r)} \quad (33)$$

Simple substitution and rearrangement yields the final results.

## 6 Back to the distribution function

We have already dealt with the distribution function in our discussion of velocity dispersion and anisotropy and it is here that Hernquist's choice to normalise the function to the mass, rather than unity, is found. In those discussions we also dealt with the fact that since the system was spherical and isotropic (i.e. described by a single-valued radial potential) the radial coordinate could be mapped directly to energy. In order to find this new function,  $f(E)$ , we need to briefly go back to earlier in the paper where I skipped over the dimensionless variables.

First, we need to understand how, mathematically, this new function fits in with what we already know and the little paragraph before equation [17] in the paper leaves out a lot of important reasoning. To begin with, recall the Hamiltonian we used in the anisotropy section and how we said that the distribution of particles was entirely dependent on that Hamiltonian. Well, that Hamiltonian was

$$H = K + V = \frac{1}{2}v^2 + \Phi(\mathbf{x}) \quad (34)$$

Now, lets deal in less absolute terms by using relative energies instead, so now we define the potential relative to some constant  $\Phi_0$ . This means that the energy of a given star will still be  $H$ , but now it will be  $H$  below the background potential  $\Phi_0$ . So, now we have

$$\Psi \equiv \Phi_0 - \Phi \text{ and } \epsilon \equiv -H + \Phi_0 = \Psi - \frac{1}{2}v^2 \quad (35)$$

In order to be consistent, we also apply the limit that as  $r \rightarrow \infty$ ,  $\Psi \rightarrow \Phi_0$  i.e. the potential at an infinite distance from the system is the same as the natural background potential.

So, in our new terms we can start by re-writing equation 11 in terms of our relative energy terms. Our  $f(\mathbf{x}, \mathbf{v})$  is given by our new energy function  $f(\epsilon)$  and when we substitute it in we now have

$$\nu(r) = \int_v f(\Psi - \frac{1}{2}v^2).d^3\mathbf{v} = 4\pi \int f(\Psi - \frac{1}{2}v^2).v^2 dv \quad (36)$$

From equation 35 we can find  $\frac{d\epsilon}{dv} = -v$ . When we collect terms and substitute for  $v$  we are left with

$$\nu(r) = 4\pi \int_0^\Psi f(\epsilon)\sqrt{2(\Psi - \epsilon)}.d\epsilon \quad (37)$$

where the  $\sqrt{2(\Psi - \epsilon)}$  term is equivalent to  $v$  and, as a fun check for the physicality of what we are doing, if you substitute  $H = K + V$  back into  $\Psi$  and  $\epsilon$  then  $v = \sqrt{2K}$  which is what we would expect. The sharp eyed among you may also have noticed that a minus sign went missing somewhere in equation 37 when we substituted in for  $v$ . This is because kinetic energy, which, as we have seen, is what we are dealing in, only cares about the magnitude of velocity, not the direction. Since we have already discussed the fact that the potential completely describes the radial component of the distribution in a spherical system we can call equation 37 a function of  $\Psi$ .

At this point we are still trying to get to the equation cited by Hernquist. The next step is to differentiate both sides w.r.t  $\Psi$ . This gives us

$$\frac{1}{\sqrt{8\pi}} \frac{d}{d\epsilon} \int_0^\Psi \frac{f(\epsilon)}{\sqrt{\Psi - \epsilon}}.d\epsilon \quad (38)$$

This just so happens to be a kind of equation known as an Abel integral.

$$f(x) = \int_0^x \frac{g(t)}{(x-t)^\alpha}.dt \rightarrow g(t) = \frac{\sin(\pi\alpha)}{\pi} \left[ \int_0^t \frac{dx}{(t-x)^{1-\alpha}} \frac{df}{dx} + \frac{f(0)}{t^{1-\alpha}} \right] \quad (39)$$

I am not going to go into the physical interpretation of these transforms here as it suffices to treat it as a purely abstract and convenient result that allows us to invert relations of the same form as equation 38 so that we can say

$$f(\epsilon) = \frac{1}{\sqrt{8\pi^2}} \left[ \int_0^\epsilon \frac{d\Psi}{\sqrt{\epsilon - \Psi}} \frac{d^2\nu}{d\Psi^2} + \frac{1}{\sqrt{\epsilon}} \frac{d\nu}{d\Psi} \Big|_{\Psi=0} \right] \quad (40)$$

This result is known as Eddington's formula.

The second-order integrals are due to our rather awkward choice of variables to match the form of the Abel formula 39. Apart from that, the above is obtained by a direct substitution from equation 38 into equation 39.

We now have a general formula relating the DF to functions of potential. In order to specify a result for the Hernquist model we need to use the equations for potential that were worked out at the start of the paper.

If you look at equation [8] in the paper you'll see that it gives a relationship between the density and dimensionless potential that follows simply enough from the definitions of the dimensionless quantities. Remember that, due to Hernquist's choice of normalising to a mass value,  $\rho(r) \equiv M\nu(r)$ . We can use the dimensionless variables defined in equations [6-8] in the paper to state that

$$\nu(r) = M\rho(r) = \frac{\tilde{\rho}}{2\pi a^3} = \frac{1}{2\pi a^3} \frac{\tilde{\Psi}^4}{1 - \tilde{\Psi}} \quad (41)$$

We need to differentiate this function twice in order to be able to substitute into equation 41. The first order of differentiation is easy enough and yields

$$\frac{d\nu}{d\Psi} = \frac{1}{2\pi a^2 GM} \frac{\tilde{\Psi}^3(4 - 3\tilde{\Psi})}{(1 - \tilde{\Psi})^2} \quad (42)$$

Evaluating the above at  $\Psi = 0$  gives zero. The prefactors come from the fact that  $\tilde{\Psi} \equiv \frac{a\Psi}{GM}$ . Now we differentiate again

$$\frac{d^2\nu}{d\Psi^2} = \frac{d}{d\tilde{\Psi}} (\tilde{\Psi}^3(4 - 3\tilde{\Psi})(1 - \tilde{\Psi})^{-2}) = \frac{a}{GM} \frac{2\tilde{\Psi}^2(6 - 8\tilde{\Psi} + 3\tilde{\Psi}^2)}{(1 - \tilde{\Psi})^3} \quad (43)$$

Now we can finally substitute back into equation 40 to bring us to the final step. We need to integrate

$$f(\epsilon) = \frac{\sqrt{2}}{(2\pi)^3(GM)^2 a} \int_0^\epsilon \frac{2\tilde{\Psi}^2(6 - 8\tilde{\Psi} + 3\tilde{\Psi}^2)}{(1 - \tilde{\Psi})^3} \frac{d\Psi}{\sqrt{\epsilon - \Psi}} \quad (44)$$

Using Maple to solve this requires, again, breaking it down into pieces as we have done before. We can easily express  $\tilde{\Psi}$  in terms of  $\Psi$  and so the integration can proceed. Eventually, we arrive at the formula given in the paper

$$\frac{1}{\sqrt{2}(2\pi)^3(GMa)^{1.5}} \frac{\sqrt{\tilde{\epsilon}}}{(1-\tilde{\epsilon})^2} [(1-2\tilde{\epsilon})(8\tilde{\epsilon}^2-8\tilde{\epsilon}-3) + \frac{3\sin^{-1}\sqrt{\tilde{\epsilon}}}{\sqrt{\tilde{\epsilon}}(1-\tilde{\epsilon})}] \quad (45)$$

where  $\tilde{\epsilon} \equiv q^2$ . In this respect the previous equation can be thought of as a function of  $E$  instead of  $\epsilon$ . The limiting cases discussed in the paper are found by various Taylor expansions. For example,

$$\sin^{-1}(x), x \rightarrow 0; \sin^{-1}(x) \rightarrow 0 \quad (46)$$

This now allows for the  $(1-q^2)$  terms to be collected and a binomial expansion to be applied

$$(1-q^2)^{-2} \equiv 1 - 2q^2 + 3q^4 - 4q^6 + \dots \quad (47)$$

Expanding the other terms in the equation and collecting terms yields the function for  $q \rightarrow 0$ . A similar process for the other limit provides that answer but the maths is equivalent i.e. using limiting cases for  $\sin^{-1}(x)$  and series expansions. The only significant change is the addition of a  $\frac{\pi}{2}$  term from the  $\sin^{-1}(1)$ .

Remember that we initially only care about the first few terms in the series since increasing powers of any number  $0 \leq n \leq 1$  will tend to 0. So, just isolate the terms that produce factors of up to  $q^4$  and discard the rest. Do the same for all the series and the coefficients should match up for those terms.

## 7 Density of States

We are reaching the end and most of the mathematical tools that we will need to complete the paper have already been used at some prior point, such as Abel formulae which will appear once more. As such, I shall not go into quite as much detail since the aim of this document is to tell what mathematical tools were used, not to suffocate under every single detail.

The next function we are interested in is the density of states,  $g(E)$ . In this context the density of states describes the volume of phase-space occupying a certain energy state. So, the mass of stars of a given binding energy is the product of the volume of phase-space occupying a given energy state and the mass occupying a given phase-space volume given as a function of energy, or equation [22] in the paper.

The density of states, as just described, is given mathematically by

$$g(E) = \int \delta(H - E) d^3\mathbf{x} d^3\mathbf{v} \quad (48)$$

This integral is then evaluated over both ranges up to some value  $E$ , or the radius and velocity that correspond to  $E$ . This means that the integral looks like

$$g(E) = (4\pi)^2 \int_0^{r(\Phi=E)} r^2 dr \int v^2 \delta\left(\frac{1}{2}v^2 + \Phi - E\right) dv \quad (49)$$

The trick here is to notice that, if you make a change of variable to  $\epsilon = \frac{v^2}{2}$  then the second integral, being a delta function, can only be evaluated at one point, where  $\epsilon = -(\Phi - E)$ . The substitution and this fact mean that the integral is now

$$g(E) = (4\pi)^2 \int_0^{\text{radius}(\Phi=E)} r^2 \sqrt{2(E - \Phi)} dr \quad (50)$$

To solve this the way suggested in Binney and Tremaine, we need to make another change of variable. Currently, we are integrating between the centre of the system and a radius at which  $\Phi = E$ . We can express this instead in terms of another coordinate we shall call  $X$ . Start from the definition of potential in equation 10 and rearrange to make  $r$  the subject

$$r = \frac{-GM}{\Phi} - a; \text{ Let } X \text{ be } \frac{-GM}{a\Phi} \rightarrow r(\Phi = E) = \frac{-GM}{E} - a = Aa - a \quad (51)$$

Substituting in  $r = aX - a$  gives us the integral we will be evaluating

$$g(E) = (4\pi)^2 \int_1^A (aX - a)^2 \sqrt{2E \left( \frac{-GM}{Aa} - \frac{-GM}{Xa} \right)} a dX = (4\pi)^2 a^3 \sqrt{2E} \int_1^A (X-1)^2 \sqrt{\left( \frac{A}{X} - 1 \right)} dX \quad (52)$$

This solves similarly to equation 44. Using Maple and putting in the limits will produce

$$(4\pi)^2 a^3 \sqrt{2E} \left[ \sqrt{A-1} \left( \frac{A^2}{8} - \frac{5A}{12} - \frac{1}{3} \right) + \frac{A}{8} (A^2 - 4A + 8) \cos^{-1}(A^{-\frac{1}{2}}) \right] \quad (53)$$

The formula in the paper is equivalent but requires substitutions for  $A = q^2$  and  $\nu_g$ . The rest of this section is treading familiar ground.  $g(E)$  has a very similar form to  $f(E)$  so the consideration of limiting cases is practically identical also. Consequently, there isn't much

point to further detail. From paper equation [22] it should be clear what we now need to do to find  $\frac{dM}{d\epsilon}$ . In order to find the limiting cases of  $\frac{dM}{d\epsilon}$  you simply find the product of the same limited cases of  $g(E)$  and  $f(E)$ . This process is long and amounts to much collection of terms and simplification. You may find it easier to multiply the two general equations together and expand the terms before applying the approximations so make it a little easier to work out what happens to the series expansions.

The steps in between were carried out using equation solvers and so detailing them at length here would not actually help your understanding of how to process these functions. If you really must know how one would solve this with pen and paper then consult Wolfram Alpha (see Acknowledgments).

## 8 Reintroducing anisotropy

The final stage in our calculation is to reintroduce the possibility that the system has varying isotropy. We are going to state that the DF depends now on another function,  $Q$ , which is

$$Q = \epsilon - \frac{L^2}{2r_a^2} \quad (54)$$

$r_a^2$  is some scale over which the anisotropy becomes apparent i.e. if  $r_a^2 = \infty$  then  $Q \equiv \epsilon$  and we get the model we have been working with up to this point as the anisotropy does not occur within  $0 \leq r \leq \infty$ . We shall also define a set of polar coordinates for dealing with velocity as follows

$$v_r = v \cos \eta; v_\theta = v \sin \eta \cos \psi; v_\phi = v \sin \eta \sin \psi \quad (55)$$

The geometry of  $\eta$  and  $\psi$  becomes apparent from the consideration that for  $\eta = 0$ ,  $v_r = v$ .  $\eta$  is the angular difference between the radial component of velocity and the radial. Similarly,  $\psi$  is the angle made by the velocity vector and a circumferential ring.

So, using these coordinates we can now write this as

$$Q = \left( \Psi - \frac{1}{2}v^2 \right) - \left( \frac{(v \cdot r \sin \eta)^2}{2r_a^2} \right) = \Psi - \frac{1}{2}v^2 \left( 1 + \frac{r^2}{r_a^2} \sin^2 \eta \right) \quad (56)$$

We are going to start by working out a familiar function, the number density  $\nu(r)$  using our new variable,  $Q$ , and the polar coordinates

$$\nu(r) = \int f(\epsilon, L).d^3\mathbf{v} = 2\pi \int_0^\pi \sin \eta. d\eta \int_0^\Psi f(Q) \frac{\sqrt{2(\Psi - Q)}}{[1 + (\frac{r}{r_a})^2 \sin^2 \eta]^{\frac{3}{2}}}. dQ \quad (57)$$

We can chose to rearrange this equation to produce an integral in  $\eta$  which happens to solve as a standard integral by substitution

$$\int_0^\pi \frac{\sin \eta}{[1 + (\frac{r}{r_a})^2 \sin^2 \eta]^{\frac{3}{2}}}. d\eta = \frac{2}{1 + (\frac{r}{r_a})^2} \quad (58)$$

So, we have

$$(1 + (\frac{r}{r_a})^2)\nu(r) = 4\pi \int_0^\Psi f(Q) \sqrt{2(\Psi - Q)}. dQ \quad (59)$$

Notice that the right hand side of this equation corresponds exactly to equation 37. We have already solved this before through the use of Abel integrals. The only difference is that now we have a modified density function

$$\nu_Q(r) \equiv (1 + (\frac{r}{r_a})^2)\nu(r) \quad (60)$$

We can now substitute our new functions  $Q$  and  $\nu_Q(r)$

$$f(\epsilon) = \frac{1}{\sqrt{8}\pi^2} \left[ \int_0^Q \frac{d\Psi}{\sqrt{Q - \Psi}} \frac{d^2\nu_Q}{d\Psi^2} + \frac{1}{\sqrt{Q}} \frac{d\nu_Q}{d\Psi} \Big|_{\Psi=0} \right] \quad (61)$$

This can be solved in precisely the same way as equation 45 using the new variables. Remember that the paper uses  $\nu(r) \equiv \frac{\rho(r)}{M}$  and that  $q$  is a constant term. For the steps to get from the Abel formula above to the result in the paper just look back at how that was accomplished earlier as the methodology is the same. The final result worth quoting here is that the model we employed here by the assumption in equation 54 is known as an Osipkov-Merrit model and they have an anisotropy parameter given by

$$\beta(r) = \frac{1}{1 + (\frac{r}{r_a})^2} \quad (62)$$

This produces ergodic behaviour for  $r \ll r_a$  and radial behaviour for  $r \gg r_a$ .

This brings us to the end of the section of the paper I am going to cover. Hopefully this has allowed you to understand the mathematics that underlies the derivations in the paper as well as the concepts and physics in it too.

## 9 Acknowledgments

The majority of the above is taken from the detailed descriptions of the same by Binney and Tremaine in Galactic Dynamics. This document mainly serves to gather the information in Galactic Dynamics and re-examine it in an order that is most applicable to the study of the Hernquist model.

Several equations were solved using a combination of Maple 13 and Wolfram Alpha. I would recommend using Maple 13 to this kind of work due to its much more readable interface but equally, if you are interested, Wolfram Alpha can tell you how it went about solving the integral. So if you need convincing that it is actually physically possible then I would recommend giving a look.

If you have any questions about this document or you spot any mistakes feel free to email me at [jab22@st-andrews.ac.uk](mailto:jab22@st-andrews.ac.uk).