Lec 4, Friday 17 Feb

- potential and eqs. of motion
  - in general geometry
  - Axisymmetric
  - spherical

Laplacian in various coordinates

Cartesians:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical:

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

<u>Example: Energy is conserved in STATIC potential</u>

- The orbital energy of a star is given by:

$$E = \frac{1}{2}v^2 + \phi(\vec{r},t)$$

$$\frac{dE}{dt} = \vec{v}\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt}\nabla\phi + \frac{\partial\phi}{\partial t} = 0 + \frac{\partial\phi}{\partial t}$$

0 since $\frac{d\vec{v}}{dt} = -\nabla\phi$ and $\frac{d\vec{r}}{dt} = \vec{v}$

0 for static potential.

So orbital Energy is Conserved dE/dt=0 only in "time-independent" potential.

Static Axisymmetric density → Static Axisymmetric potential

- We employ a cylindrical coordinate system (R, ϕ,z) e.g., centred on the galaxy and align the z axis with the galaxy axis of symmetry.
- Here the potential is of the form ϕ(R,z).
- Density and Potential are Static and Axisymmetric
  - independent of time and azimuthal angle

$$\phi(R,z) \Rightarrow \rho(R,z) = \frac{1}{4\pi G}\left[R\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right) + \frac{\partial^2\phi}{\partial z^2}\right]$$

$$g_r = -\frac{\partial\phi}{\partial R} \qquad g_z = -\frac{\partial\phi}{\partial z}$$

<u>Orbits in an axisymmetric potential</u>

- Let the potential which we assume to be symmetric about the plane z=0, be ϕ(R,z).
- The general equation of motion of the star is

$$\frac{d^2\vec{r}}{dt^2} = -\nabla\phi(R,z) \qquad \text{Eq. of Motion}$$

- Eqs. of motion in cylindrical coordinates

$$\ddot{z} = -\frac{\partial\phi}{\partial z}, \quad \ddot{R} - R\dot{\theta}^2 = -\frac{\partial\phi}{\partial R}, \quad 2\dot{R}\dot{\theta} + R\ddot{\theta} = \frac{d}{Rdt}(R^2\dot{\theta}) = -\frac{\partial\phi}{R\partial\theta} = 0$$

Conservation of angular momentum z-component Jz if axisymmetric

$$J_Z = R^2\dot{\theta} \Rightarrow \frac{d}{dt}Jz = \frac{d}{dt}(R^2\dot{\theta}) = 0$$

- The component of angular momentum about the z-axis is conserved.
- If ϕ(R,z) has no dependence on θ then the azimuthal angular momentum is conserved
  - or because z-component of the torque <u>r</u>×<u>F</u>=0. (Show it)

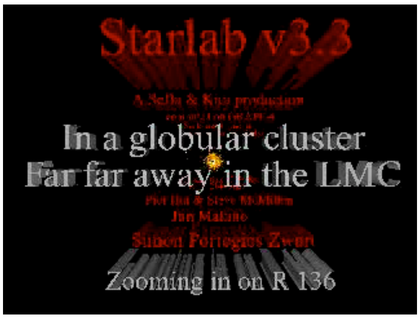
Spherical Static System

- Density, potential function of radius |r| only
- Conservation of
  - energy E,
  - angular momentum J (all 3-components)
  - Argue that a star moves orbit which confined to a plane perpendicular to J vector.

Spherical Cow Theorem

- Most astronomical objects can be approximated as spherical.
- Anyway non-spherical systems are too difficult to model, almost all models are spherical.

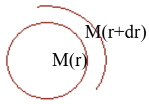
Globular: A nearly spherical static system



From Spherical Density to Mass

$$\mathrm{M(R+dr) = M(R) + dM}$$

$$\mathrm{dM} = \rho(\mathrm{r})\mathrm{d}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \rho(\mathrm{r}) dr$$



$$\rho(r) = \frac{dM}{d\left(\frac{4}{3}\pi r^3\right)} = \frac{dM}{4\pi r^2 dr}$$

$$M(R) = \int \rho d\left(\frac{4}{3}\pi r^3\right)$$

Theorems on Spherical Systems

- NEWTONS 1st THEOREM:A body that is inside a spherical shell of matter experiences no net gravitational force from that shell
- NEWTONS 2nd THEOREM:The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the matter were concentrated at its centre.

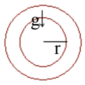
Poisson's eq. in Spherical systems

- Poisson's eq. in a spherical potential with no θ or Φ dependence is:

$$\nabla^2 \varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = 4\pi G \rho(r)$$

Interpretation of Poissons Equation

- Consider a spherical distribution of mass of density ρ(r).



$$g = -\frac{GM(r)}{r^2}$$

$$\phi = \int_r^\infty g(r)dr \qquad \text{since } \phi = 0 \text{ at } \infty \text{ and is} < 0 \text{ at r}$$

$$= -\int_r^\infty \frac{GM(r)}{r^2}dr$$

$$\text{Mass Enclosed} = \int_r^\infty 4\pi r^2 \rho(r)dr$$

- Take d/dr and multiply r^2 →

$$r^2\frac{d\phi}{dr} = -gr^2 = GM(r) = \left(G\int 4\pi r^2 \rho(r)dr\right)$$

- Take d/dr and divide r^2→

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2 g\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(GM) = 4\pi G\rho(r)$$

$$\rightarrow \nabla^2\phi = -\vec{\nabla}.g = 4\pi G\rho$$

Escape Velocity

- ESCAPE VELOCITY= velocity required in order for an object to escape from a gravitational potential well and arrive at ∞ with zero KE.

=0 often

$$\phi(r) = \phi(\infty) - \frac{1}{2}v_{esc}^2$$

$$\rightarrow v_{esc}(r) = \sqrt{2\phi(\infty) - 2\phi(r)}$$

Plummer Model for star cluster

- A spherically symmetric potential of the form:

$$\phi = -\frac{GM}{\sqrt{r^2 + a^2}}$$

e.g., for a globular cluster a=1pc, M=10^5 Sun Mass show Vesc(0)=30km/s

- Show corresponding to a density (use Poisson's eq):

$$\rho = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{5}{2}}$$

What have we learned?

- Conditions for conservation of orbital energy, angular momentum of a test particle
- Meaning of escape velocity
- How Poisson's equation simplifies in cylindrical and spherical symmetries

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Lec 5, Tue 21 Feb

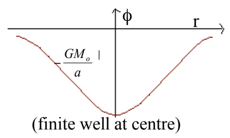
A worked-out example: Hernquist Potential for stars in a galaxy

$$\phi_*(r) = -\frac{GM_0}{a+r}$$, use Poisson eq. show

$$\rho_*(r) = \frac{M_0}{2\pi a^3}\left(\frac{r}{a}\right)^{-1}\left(1+\frac{r}{a}\right)^{-2}$$

- E.g., a=1000pc, $M_0=10^{10}$ solar, show central escape velocity Vesc(0)=300km/s,
- Show M_0 has the meaning of total mass
  - Potential at large r is like that of a point mass M_0
  - Integrate the density from r=0 to inifnity also gives M_0

Potential of globular clusters and galaxies looks like this:



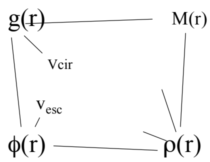
$\phi(0) = -const$ (finite well at centre)

$\phi(r) \propto r^{-1}$ (Kepler for large r)

→ *Centre* is the minimum of potential with escape velocity

$$v_{esc}(0) = \sqrt{\frac{2GM_o}{a}}$$

Links between dynamical quantities



Circular Velocity

- CIRCULAR VELOCITY= the speed of a test particle in a circular orbit at radius r.

$$|g| = \frac{v_{cir}^2}{r} = \nabla\phi = \frac{GM(r)}{r^2}$$

$$\Rightarrow M(r) = \frac{v_{cir}^2 r}{G}$$

For a point mass M: Show in a uniform density sphere

$$v_c(r) = \sqrt{\frac{GM}{r}} \qquad v_c(r) = \sqrt{\frac{4\pi G\rho}{3}}r \quad \text{since } M(r) = \frac{4}{3}\pi r^3\rho$$

What have we learned?

- How to apply Poisson's eq.
- How to relate
  - Vesc with potential and
  - Vcir with gravity
- The meanings of
  - the potential at very large radius,
  - The enclosed mass

Lec 6, Fri, 24 Feb

Motions in spherical potential

Equation of motion

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = \vec{g} = -\nabla\phi$$

If spherical

$$g_r = -\frac{\partial\phi}{\partial r}$$

If no gravity

$$\mathbf{x}(t) = \mathbf{v}_0 t + \mathbf{x}_0$$

$$\mathbf{v}(t) = \mathbf{v}_0$$

Conserved if spherical static

$$E = \frac{1}{2}v^2 + \phi(r)$$

$$L = J = \mathbf{x} \otimes \mathbf{v} = r v_t \cdot \hat{n}$$

Proof: Angular Momentum is Conserved if spherical

- $\vec{L} = \vec{r} \times \vec{v}$

→ $$\frac{d\vec{L}}{dt} = \frac{d(\vec{r}\times\vec{v})}{dt} = \frac{d\vec{r}}{dt}\times\vec{v} + \vec{r}\times\frac{d\vec{v}}{dt} = 0 + \vec{r}\times\vec{g}$$

Since $\frac{\partial\phi}{\partial\theta} = 0$ then the spherical force g is in the r direction, no torque

→both cross products on the RHS = 0.

So Angular Momentum L is Conserved

$$\frac{d\vec{L}}{dt} = 0$$

In static spherical potentials: star moves in a plane (r,θ)

- central force field $\underline{g} = g\hat{r}$
- angular momentum $\underline{r} \times \dot{\underline{r}} = \underline{L}$
- equations of motion are
  - radial acceleration: $\ddot{r} - r\dot{\theta}^2 = g(r)$
  - tangential acceleration: $2\dot{r}\dot{\theta} + r\ddot{\theta}^2 = 0$

$$r^2\dot{\theta} = \text{constant} = L$$

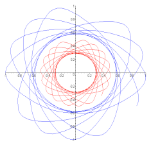
Orbits in Spherical Potentials

- The motion of a star in a centrally directed field of force is greatly simplified by the familiar law of conservation (WHY?) of angular momentum.

$$\vec{L} = \vec{r} \times \frac{d\vec{r}}{dt} = \text{const}$$

$$= r^2\frac{d\theta}{dt} = 2\frac{\text{area swept}}{\text{unit time}}$$

Keplers 3rd law



Energy Conservation (WHY?)

$$E = \phi(r) + \frac{1}{2}\left(\frac{d\vec{r}}{dt}\right)^2 + \frac{1}{2}\left(r\frac{d\theta}{dt}\right)^2$$

ϕ_{eff}

$$= \phi(r) + \frac{L^2}{2r^2} + \frac{1}{2}\left(\frac{d\vec{r}}{dt}\right)^2$$

$$\frac{dr}{dt} = \pm\sqrt{2E - 2\phi_{eff}(r)}$$

Orbit in the z=0 plane of a disk potential φ(R,z).

- Energy/angular momentum of star (per unit mass)

$$E = \tfrac{1}{2}\left[\dot{R}^2 + (R\dot{\theta})^2\right] + \Phi(R,0)$$

$$= \tfrac{1}{2}\left[\dot{R}^2\right] + \left[\frac{J_z^2}{2R^2} + \Phi(R,0)\right]$$

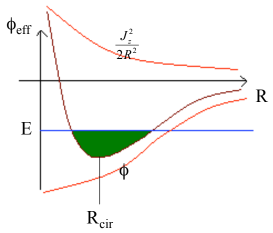
$$= \tfrac{1}{2}\left[\dot{R}^2\right] + \Phi_{\text{eff}}(R,0)$$

- orbit bound within

$$\Rightarrow E \geq \Phi_{\text{eff}}(R,0)$$

Radial Oscillation

- An orbit is bound between two radii: a loop
- Lower energy E means thinner loop (nearly circular closed) orbit

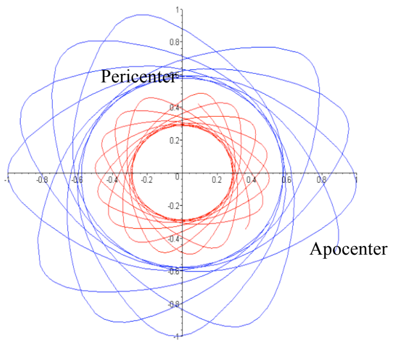


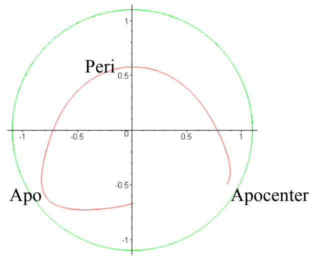
Eq of Motion for planar orbits

- EoM:

$$\ddot{R} = -\frac{\partial \Phi_{eff}}{\partial R}; \qquad z=0$$

If circular orbit R=cst, $z = 0$ => $\frac{\partial \Phi_{eff}}{\partial R} = 0$ at $R = R_c$





Apocenter and pericenter

- No radial motion at these turn-around radii
  - $dr/dt = V_r = 0$ at apo and peri
- Hence
- $J_z = R\, V_t$
  $= R_a V_a = R_p V_p$
- $E = \frac{1}{2}(V_r^2 + V_t^2) + \Phi(R,0) = \frac{1}{2} V_r^2 + \Phi_{eff}(R,0)$
  $= \frac{1}{2} V_a^2 + \Phi(R_a,0)$
  $= \frac{1}{2} V_p^2 + \Phi(R_p,0)$

Orbit in axisymmetric disk potential ϕ (R,z).

- Energy/angular momentum of star (per unit mass)

$$E = \tfrac{1}{2}\left[\dot{R}^2 + \left(R\dot{\theta}\right)^2 + \dot{z}^2\right] + \Phi$$

$$= \tfrac{1}{2}\left[\dot{R}^2 + \dot{z}^2\right] + \frac{J_z^{\ 2}}{2R^2} + \Phi$$

$$= \tfrac{1}{2}\left[\dot{R}^2 + \dot{z}^2\right] + \Phi_{eff}(R,z)$$

- orbit bound within

$$\Longrightarrow E \geq \Phi_{eff}(R,z)$$

EoM for nearly circular orbits

- EoM: $\ddot{R} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial R}; \qquad \ddot{z} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial z}$
- Taylor expand

$$\frac{\partial \Phi_{\mathrm{eff}}}{\partial R} = \frac{\partial \Phi_{\mathrm{eff}}}{\partial z} = 0 \quad \text{at} \quad R = R_g, z = 0$$

$$\Phi_{\mathrm{eff}} = \tfrac{1}{2}\left(\frac{\partial^2 \Phi_{\mathrm{eff}}}{\partial R^2}\right)_{(R_g,0)} x^2 + \tfrac{1}{2}\left(\frac{\partial^2 \Phi_{\mathrm{eff}}}{\partial z^2}\right)_{(R_g,0)} z^2 + \dots$$
$$= \kappa^2 x^2/2 + \quad \nu^2 z^2/2 + \dots$$

– $x = R - R_g$

Sun's Vertical and radial epicycles

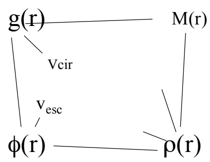
- harmonic oscillator +/-10pc every 10^8 yr

κ – epicyclic frequency :

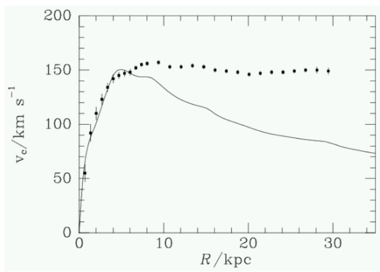
ν – vertical frequency :

$$\ddot{R} = -\kappa^2 R, \quad \text{and} \quad \ddot{z} = -\nu^2 z$$

Links between dynamical quantities



Stars are not enough: add Dark Matter in galaxies



NGC 3198 (Begeman 1987)

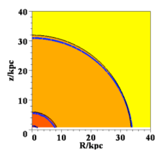
Bekenstein & Milgrom (1984)
Bekenstein (2004), Zhao & Famaey (2006)

- Modify gravity g,
  – Analogy to E-field in medium of varying Dielectric

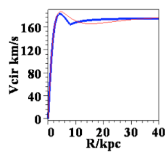
$$-\nabla \cdot \left(\frac{g}{4\pi \overline{G}}\right) = \rho_*(r)$$
$$\overline{G}(g/a_0) = (1 + a_0/g)\, G$$
$$\sim G \qquad \text{if } g = |\nabla\phi| > a_0$$
$$\sim G\, a_0/g > G \qquad \text{if } g < a_0$$

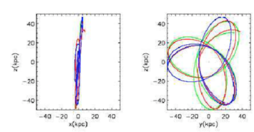
- Gradient of Conservative potential

MOND similar to DM in potential, rotation curve, orbit



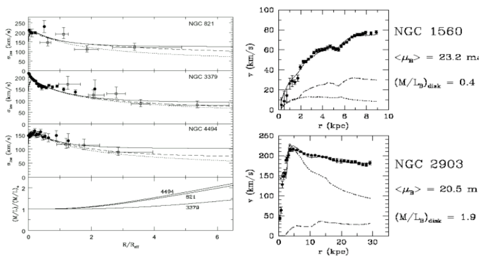
Zhao (2005)





Read & Moore (2005)

Explained: Fall/Rise/wiggles in Ellip/Spiral/Dwarf galaxies



Milgrom & Sanders, Sanders & McGaugh

What have we learned?

- Orbits in a spherical potential or in the mid-plane of a disk potential
- How to relate Pericentre, Apocentre through energy and angular momentum conservation.
- Rotation curves of galaxies
  - Need for Dark Matter or a boosted gravity

Tutorial: Singular Isothermal Sphere

- Has Potential Beyond r_o: $\phi(r) = -\frac{GM_0}{r}$
- And Inside $r<r_0$ $\phi(r) = v_0^2 \ln\frac{r}{r_o} + \phi_o$
- Prove that the potential AND gravity is continuous at $r=r_o$ if $\phi_0 = -GM_0/r_0 = -v_0^2$
- Prove density drops sharply to 0 beyond r0, and inside r0

$$\rho(r) = \frac{V_0^2}{4\pi G r^2}$$

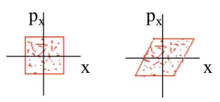
- Integrate density to prove total mass=M0
- What is circular and escape velocities at r=r0?
- Draw diagrams of M(r), Vesc(r), Vcir(r), |Phi(r)|, rho(r), |g(r)| vs. r (assume V0=200km/s, r0=100kpc).

Another Singular Isothermal Sphere

- Consider a potential $\Phi(r)=V_0^2\ln(r)$.
- Use Jeans eq. to show the velocity dispersion σ (assume isotropic) is constant V_0^2/n for a spherical tracer population of density $A*r^{-n}$; Show we required constants $A = V_0^2/(4*Pi*G)$. and n=2 in order for the tracer to become a self-gravitating population. Justify why this model is called Singular Isothermal Sphere.
- Show stars with a phase space density $f(E)= \exp(-E/\sigma^2)$ inside this potential well will have no net motion $\langle V\rangle=0$, and a constant rms velocity σ in all directions.
- Consider a black hole of mass m on a rosette orbit bound between pericenter r_0 and apocenter $2r_0$. Suppose the black hole decays its orbit due to dynamical friction to a circular orbit $r_0/2$ after time t_0. How much orbital energy and angular momentum have been dissipated? By what percentage has the tidal radius of the BH reduced? How long would the orbital decay take for a smaller black hole of mass m/2 in a small galaxy of potential $\Phi(r)=0.25V_0^2\ln(r)$. ? Argue it would take less time to decay from r_0 to r_0 /2 then from $r_0/2$ to 0.

Incompressible df/dt=0

- N_{star} identical particles moving in a small bundle in phase space (Vol=$\Delta_x \Delta_p$),
- phase space deforms but maintains its area.
  - Likewise for y-p_y and z-p_z.



$\frac{dVol}{d\lambda} = 0, \frac{dNstar}{d\lambda} = 0,$ 'LIOUVILLES THEOREM'

Phase space density f=Nstars/$\Delta_x \Delta_p$ ~ const

Stars flow in phase-space

- Flow of points in phase space ~ stars moving along their orbits.
- phase space coords: $(\underline{x},\underline{v}) \equiv \underline{w} \equiv (w_1, w_2, ..., w_6)$

$$\underline{\dot{w}} = (\underline{\dot{x}}, \underline{\dot{v}}) = (\underline{v}, -\nabla\Phi)$$

Collisionless Boltzmann Equation

- Collisionless df/dt=0:

$$\frac{\mathrm{d}}{\mathrm{dt}} f(x, v, t) = \left(\frac{\partial}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial}{\partial w_\alpha} \right) f(w, t) = 0$$

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left[v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] = 0$$

- Vector form

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \underline{v}} = 0$$