Steady-State Accretion Disc

- thickness $\frac{H}{R} \approx \frac{c_S}{V_\phi}$ $\quad V_\phi = \left(\frac{G\,M}{R}\right)^{1/2}$
- surface density $\Sigma \equiv \int_{-\infty}^{\infty} \rho \; dz = \sqrt{2\pi}\; \rho_0 \; H$
  - (kg/m^2)
- accretion rate $\dot{M} = 2\pi\; R\; (-V_R)\; \Sigma$
  - (kg / s)

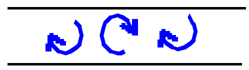
$$= 3\pi\; \nu\; \Sigma \left(1 - \left(\frac{R_*}{R}\right)^{1/2} \right)$$

- inflow velocity $V_R \approx -\frac{3\,\nu}{2\,R}$
- viscosity $\nu \equiv \alpha\;\; c_S \;\; H$
  - (m^2 / s)
  - -- alpha model (hides uncertain physics)

Anomalous Viscosity

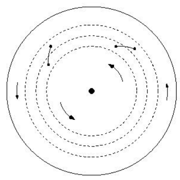
- Viscosity arises from turbulent eddies
- largest eddie size ~ H
- largest eddie velocity ~ sound speed

$$\nu = \alpha\; c_S\; H$$



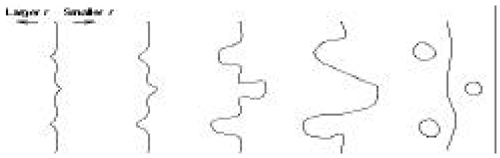
- alpha = dimensionless scale factor
- 0 < alpha < 1

Magneto-Rotational Instability

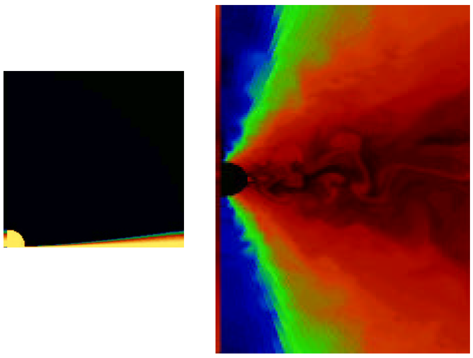


Magnetic fields link different annuli

generate MHD turbulence



Magneto-Rotational Instability



Timescales

- Dynamical timescale -- Kepler orbit time

$$t_{dyn} \sim \frac{R}{V_\phi} \sim \Omega^{-1}$$

- Vertical hydrostatic equilibrium timescale

$$t_{hse} \sim \frac{H}{c_S} \sim \Omega^{-1}$$

- Radial inflow time

$$t^{visc} \sim \frac{R}{V_R} \sim \frac{R^2}{\nu} \sim \frac{R^2}{\alpha\, c_S\, H} \sim \alpha^{-1} \left(\frac{R}{H}\right)^2 \Omega^{-1}$$

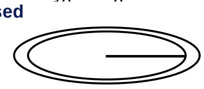
Steady-State Disc T(R)

- Energy (kinetic + potential) per mass

$$e = \frac{V_\phi^2}{2} + \Phi^{grav} = \frac{G\,M}{2R} - \frac{G\,M}{R} = -\frac{1}{2}\frac{G\,M}{R} \qquad \frac{de}{dR} = \frac{1}{2}\frac{G\,M}{R^2}$$

- energy radiated = energy released

$$\sigma\, T^4 \times 2\pi\, R\, \Delta R \times 2 = \dot{M}\, \frac{de}{dR}\, \Delta R$$



- Temperature profile

$$T^4 = \frac{G\,M\,\dot{M}}{8\pi\sigma R^3} \qquad T \propto \left(M\,\dot{M}\right)^{1/4} R^{-3/4}$$

- Disc Luminosity

$$L_{disk} = 2\int_{R_*}^{\infty} 2\pi\; R\, dR\, \sigma\; T^4 = \frac{G\,M\,\dot{M}}{2}\int_{R_*}^{\infty} \frac{dR}{R^2} = \frac{G\,M\,\dot{M}}{2\,R_*}$$

Note: 1/2 of energy remains as kinetic energy

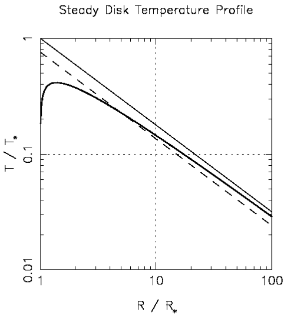
Temperature Profile

- **Include work done by viscous torques**

$$T^4 = \frac{3GM\dot{M}}{8\pi\sigma R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

$$= T_*^4\left(\frac{R}{R_*}\right)^{-3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

$$\boxed{T_* = \left(\frac{3GM\dot{M}}{8\pi\sigma R_*^3}\right)^{1/4}}$$



Inner disc luminosity lower.

Outer disk luminosity 3x higher.

max T occurs outside min R

Maximum Temperature

- **Including work done by viscous torques**

$$\left(\frac{T}{T_*}\right)^4 = \left(\frac{R}{R_*}\right)^{-3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right] \qquad T_* = \left(\frac{3GM\dot{M}}{8\pi\sigma R_*^3}\right)^{1/4}$$

- **Maximum temperature**

$$\frac{d}{dR}\left[T^4\right] = T_*^4\,\frac{d}{dR}\left[\frac{1}{R^3}-\frac{R_*^{1/2}}{R^{5/2}}\right]$$

$$= T_*^4\left[-\frac{3}{R^4}+\frac{5}{2}\frac{R_*^{1/2}}{R^{7/2}}\right] \rightarrow \boxed{\frac{R}{R_*} = \left(\frac{6}{5}\right)^2 = 1.44}$$

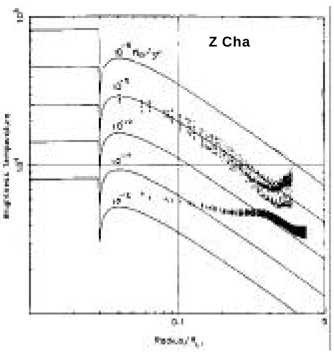
$$\boxed{\frac{T_{max}}{T_*} = \left(\frac{5}{6}\right)^{3/2}\left(1-\frac{5}{6}\right)^{1/4} = \frac{5^{3/2}}{6^{7/4}} \cong 0.486}$$

Observed Temperature Profile

Eclipse lightcurve shapes used to map temperature profile of disk.

Outburst disk matches steady disk theory

Quiescent disk does not (optically thin)



Disc Luminosity

- **Including work done by viscous torques**

$$T^4 = \frac{3GM\dot{M}}{8\pi\sigma R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

- **Disc luminosity** $L_{disk} = 2\int_{R_*}^{\infty} 2\pi R\, dR\, \sigma T^4$

$$= \frac{GM\dot{M}}{2}\int_{R_*}^{\infty} dR\left(\frac{1}{R^2}-\frac{R_*^{1/2}}{R^{5/2}}\right)$$

$$= \frac{GM\dot{M}}{2}\left[-\frac{1}{R}+\frac{3R_*^{1/2}}{2R^{3/2}}\right]_{R_*}^{\infty}$$

$$= \frac{GM\dot{M}}{2R_*}$$

Same as before.

Energy is conserved but re-distributed in radius by viscous torques.

Blackbody Disc Spectrum

$$F_\nu = \int B_\nu(T(R))\,\frac{2\pi R\, dR\cos i}{D^2}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2\left(e^x-1\right)} = \frac{2k^3T^3}{c^2h^2}\frac{x^3}{e^x-1} \qquad x \equiv \frac{h\nu}{kT}$$

$$\frac{T}{T_*} = \left(\frac{R}{R_*}\right)^{-3/4} = \left(\frac{x}{x_*}\right)^{-1} \rightarrow B_\nu(\lambda, T) = \frac{2k^3T_*^3}{c^2h^2}\frac{x_*^3}{e^x-1}$$

$$\frac{R}{R_*} = \left(\frac{x}{x_*}\right)^{4/3} \qquad \frac{dR}{R_*} = \frac{4}{3}\left(\frac{x}{x_*}\right)^{1/3}\left(\frac{dx}{x_*}\right)$$

Blackbody Disc Spectrum

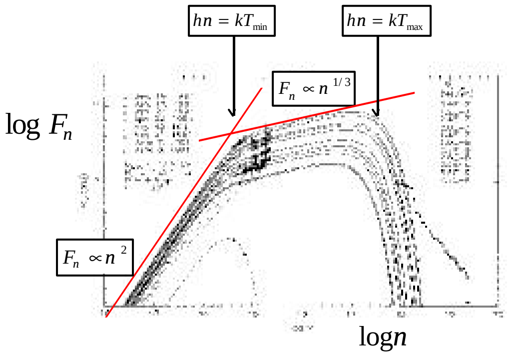
$$F_\nu = \int_{R_*}^{\infty} B_\nu(T(R))\,\frac{2\pi R\, dR\cos i}{D^2}$$

$$= \int_{x_*}^{\infty} \frac{2\left(kT_*\right)^3}{\left(ch\right)^2}\frac{x_*^3}{e^x-1}\frac{2\pi R_*^2\cos i}{D^2}\frac{4}{3}\left(\frac{x}{x_*}\right)^{5/3}\left(\frac{dx}{x_*}\right)$$

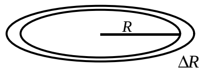
$$= \frac{16\pi R_*^2\cos i}{3D^2}\frac{\left(kT_*\right)^3}{\left(hc\right)^2}\left(\frac{h\nu}{kT_*}\right)^{1/3}\int_{x_*}^{\infty}\frac{x^{5/3}dx}{e^x-1}$$

$$\boxed{F_\nu \propto \nu^{1/3}}$$

Blackbody Disc Spectrum



Surface Density Evolution



$\Delta m = 2p\ R\ \Delta R\ \Sigma$

$\Delta \ell = 2p\ R\ \Delta R\ \Sigma\ R^2\ \Omega$

mass conservation :

$$\frac{\partial}{\partial t}\left(2p\ R\,\Delta R\,\Sigma\right) = V_R\ 2p\ R\ \Sigma\Big|_{R+\Delta R}^{R}$$

$$R\ \dot{\Sigma} + \frac{\partial}{\partial R}\left(\ R\,\Sigma\,V_R\ \right) = 0$$

angular momentum :

$$R\,\frac{\partial}{\partial t}\left(\ \Sigma\,R^2\ \Omega\ \right) + \frac{\partial}{\partial R}\left(\ R\,\Sigma\,V_R\ R^2\ \Omega\ \right) = \frac{1}{2p}\frac{\partial G}{\partial R}$$

viscous torque : $\quad G(R,t) = 2p\ \ R n\ \Sigma\,R^2\ \frac{\partial \Omega}{\partial R}$

surface density evolution (diffusion) :

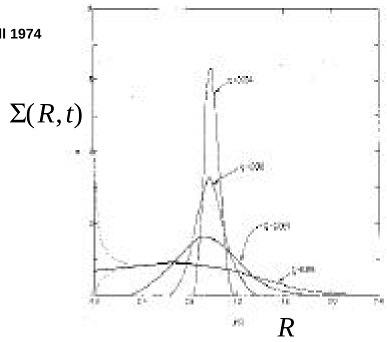
$$\dot{\Sigma} = \frac{3}{R}\frac{\partial}{\partial R}\left(\ R^{1/2}\frac{\partial}{\partial R}\left[n\ \Sigma\,R^{1/2}\ \right]\right)$$

simple diffusion:

$$\dot{\Sigma} = n\ \frac{\partial^2 \Sigma}{\partial\ R^2}$$

Ring spreads to form a disc

Pringle, Lynden-Bell 1974
MNRAS 168 603



Surface density evolution

diffusion in radius