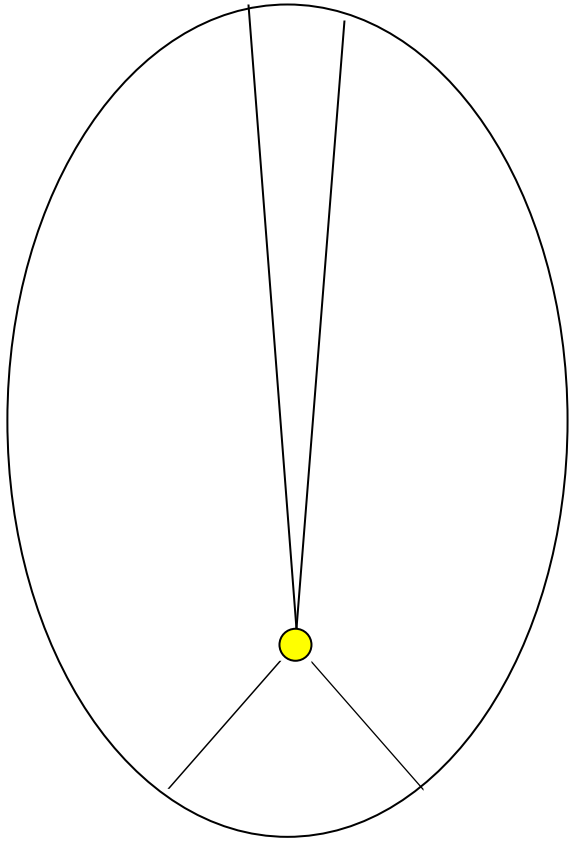
Kepler -- kinematics

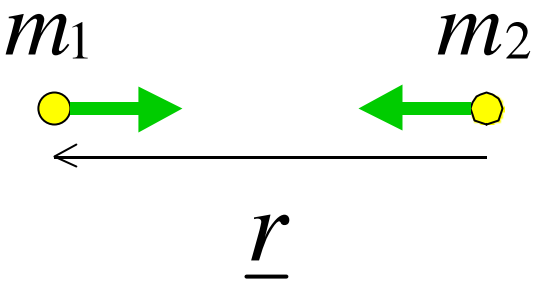
- **Empirical fit to Tycho Brahe's planet motion observations**

- **Kepler's laws:**
- **1. Elliptical orbits**
  Sun at focus
- **2. Equal areas in equal times**
- **3.** $P^2 \propto a^3$



Newton -- dynamics

$$\underline{F}_1 = -\underline{F}_2$$



$$\underline{F} = m\,\ddot{\underline{r}}$$

$$\underline{F}_1 = -\frac{G\,m_1\,m_2}{r^2}\,\hat{\underline{r}}$$

The *N*-body problem

$$\Phi(\underline{r},t) = -\sum_{i=1}^{N} \frac{G\,m_i}{|\underline{r}-\underline{r}_i|} = \text{gravitational potential}$$

$$\underline{f} = -\underline{\nabla}\Phi = \text{acceleration}$$

$$\text{equation of motion}: \quad m\,\underline{\ddot{r}} = \underline{F} = m\,\underline{f} = -m\,\underline{\nabla}\Phi$$

$$\underline{J} \equiv \underline{r}\times(m\,\underline{\dot{r}}) = \text{angular momentum}$$

$$\underline{\dot{J}} = \underline{r}\times(m\,\underline{\ddot{r}}) + \underline{\dot{r}}\times(m\,\underline{\dot{r}})$$

$$= \underline{r}\times\underline{F} \equiv \underline{G} = \text{torque} \to 0$$

Note: If $\underline{F} \parallel \underline{r}$ then $\underline{J} = \text{constant}$

and $\underline{r}$, $\underline{\dot{r}}$ (both $\perp \underline{J}$) define orbital plane.

Energy of orbit

Kinetic Energy : $K \equiv \frac{1}{2} m \left| \underline{\dot{r}} \right|^2 = \frac{1}{2} m (\underline{\dot{r}} \bullet \underline{\dot{r}})$

$$\dot{K} = m \underline{\dot{r}} \bullet \underline{\ddot{r}} = \underline{\dot{r}} \bullet \underline{F} = -m \underline{\dot{r}} \bullet \underline{\nabla} \Phi$$

Potential Energy : $W \equiv m \Phi$

$$\dot{W} = m \dot{\Phi} + m \left(\frac{d\Phi}{dx} \frac{dx}{dt} + \frac{d\Phi}{dy} \frac{dy}{dt} + \frac{d\Phi}{dz} \frac{dz}{dt} \right)$$

$$= m \dot{\Phi} + m \nabla \Phi \bullet \underline{\dot{r}}$$

Total Energy : $E = K + W$

$$\dot{E} = \dot{K} + \dot{W} = \left(- m \underline{\dot{r}} \bullet \nabla \Phi \right) + \left(m \dot{\Phi} + m \nabla \Phi \bullet \underline{\dot{r}} \right) = m \dot{\Phi}$$

$\therefore$ Energy is constant if $\dot{\Phi} = 0$

Orbits in spherical potentials

If $\Phi(r)$ depends on $|\underline{r}|$, not $\underline{r}$, then

$$\underline{f} = -\underline{\nabla}\Phi = -\frac{d\Phi}{dr}\hat{\underline{r}}$$

then

$$\dot{\underline{J}} = \underline{r} \times m\underline{f} = -m\frac{d\Phi}{dr}\underline{r} \times \hat{\underline{r}} = 0$$

$\rightarrow$ angular momentum about origin is conserved !

$\therefore$ orbital motion is in a plane

$\underline{r}$, $\dot{\underline{r}}$ in plane,

specific angular momentum

$\underline{L} \equiv \underline{J}/m$ perpendicular to plane

Kinematics

position $\underline{r}(t)$

velocity $\dot{\underline{r}} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$

speed $V = \dot{r}^2 + r^2\,\dot{\theta}^2$

unit vectors rotate with θ along the trajectory

$$\frac{d\hat{\underline{r}}}{dt} = \frac{d\hat{\underline{r}}}{d\theta}\frac{d\theta}{dt} = \hat{\underline{\theta}}\,\dot{\theta}$$

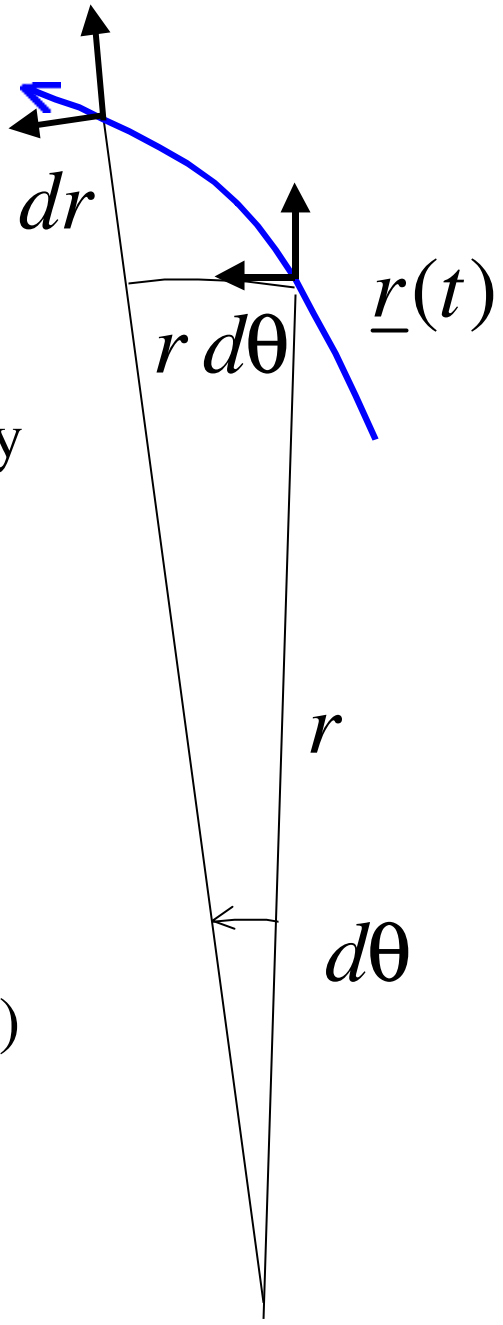
$$\frac{d\hat{\underline{\theta}}}{dt} = \frac{d\hat{\underline{\theta}}}{d\theta}\frac{d\theta}{dt} = -\hat{\underline{r}}\,\dot{\theta}$$

acceleration

$$\ddot{\underline{r}} = \ddot{r}\,\hat{\underline{r}} + \dot{r}\,(\dot{\theta}\,\hat{\underline{\theta}}) + \dot{r}\,\dot{\theta}\,\hat{\underline{\theta}} + r\,\ddot{\theta}\,\hat{\underline{\theta}} + r\,\dot{\theta}\,(-\dot{\theta}\,\hat{\underline{r}})$$

$$= \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{\underline{r}} + \left(2\,\dot{r}\,\dot{\theta} + r\ddot{\theta}\right)\hat{\underline{\theta}}$$

should be 0 if $\underline{F}$ parallel to $\underline{r}$



Kepler's 2nd Law

area swept out by $\underline{r}$ in time dt

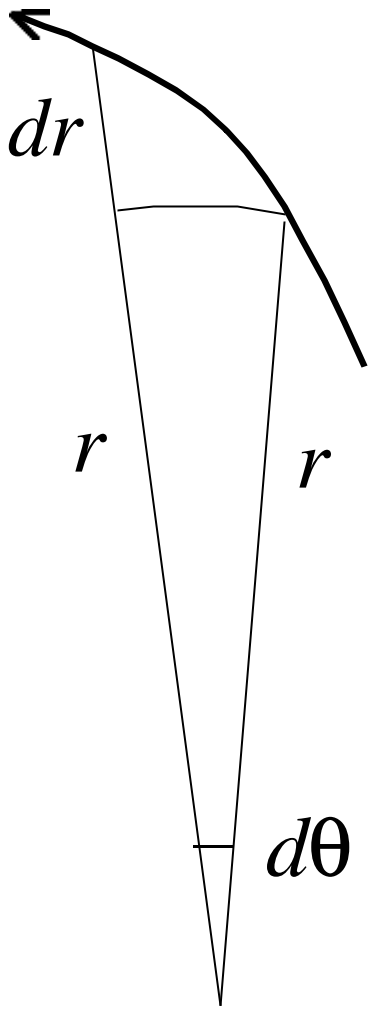
$$dA = \tfrac{1}{2}\left(r^2\, d\theta + r\, dr\, d\theta\right)$$

$$\dot{A} = \tfrac{1}{2} r^2 \dot{\theta}$$

$$= \tfrac{1}{2}\underline{r} \times \dot{\underline{r}} = \frac{J}{2m} = \text{constant}$$

$$\text{Note}: \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

transverse component of acceleration $= 0$



Conic Sections

$$r = \frac{\ell}{1 + e\cos\theta}$$

e = eccentricity

> 1 hyperbola

$= 1$ parabola

< 1 ellipse

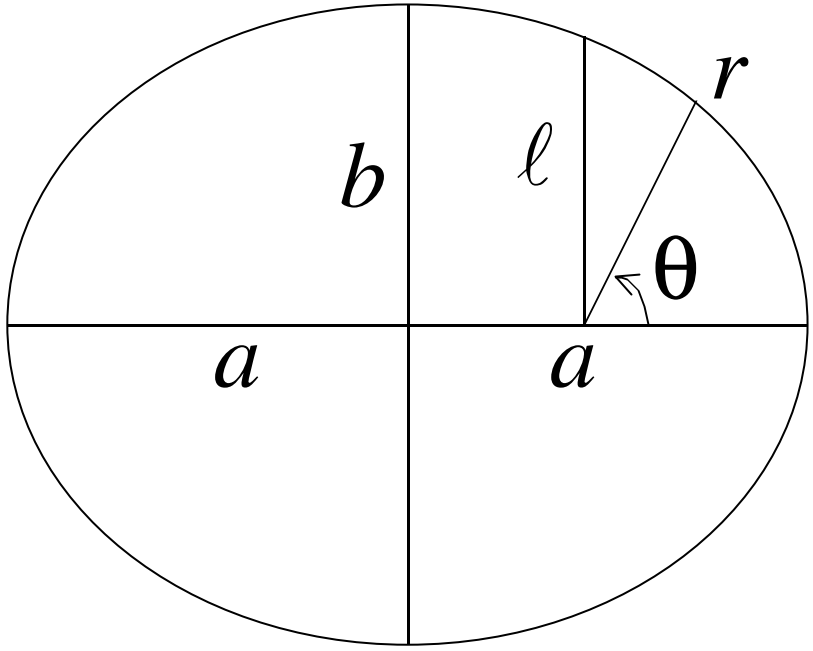
$= 0$ circle

ℓ = semi - latus rectum

a = semi - major axis

b = semi - minor axis

$$= a\left(1 - e^2\right)^{1/2}$$



$\theta = 0$	$r = \frac{\ell}{1+e}$	$= a(1-e)$	periastron
$\theta = 90^\circ$	$r = \ell$	$= a(1-e^2)$	
$\theta = 180^\circ$	$r = \frac{\ell}{1-e}$	$= a(1+e)$	apastron