Radial Potential

energy and angular momentum :

$$\varepsilon \equiv \frac{E}{m} = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{G\,M}{r}$$

$$L \equiv \frac{J}{m} = r^2\dot{\theta}$$

effective potential : $\varepsilon = \frac{\dot{r}^2}{2} + \Phi(r)$

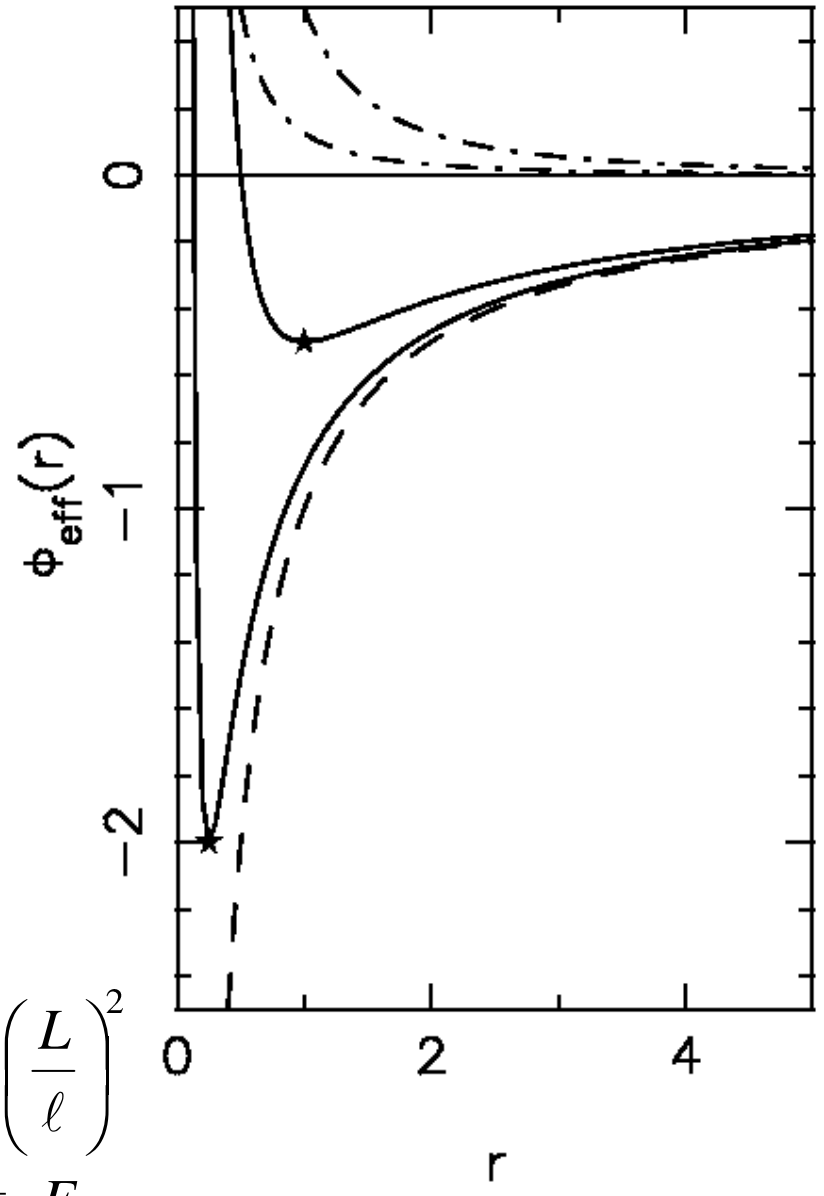
$$\Phi(r) = \frac{L^2}{2\,r^2} - \frac{G\,M}{r} = \varepsilon_o\left(\frac{\ell^2}{r^2} - 2\,\frac{\ell}{r}\right)$$

circular orbit :

$$\frac{\partial\,\Phi}{\partial\,r} = \frac{G\,M}{r^2} - \frac{L^2}{r^3} = 0 \rightarrow r = \frac{L^2}{G\,M} \equiv \ell$$

$$\Phi(\ell) = -\varepsilon_o \quad \varepsilon_o \equiv \frac{G\,M}{2\,\ell} = \frac{1}{2}\left(\frac{G\,M}{L}\right)^2 = \frac{1}{2}\left(\frac{L}{\ell}\right)^2$$

Note : ℓ, ε_o depend on M and J but not E.



Types of Orbits

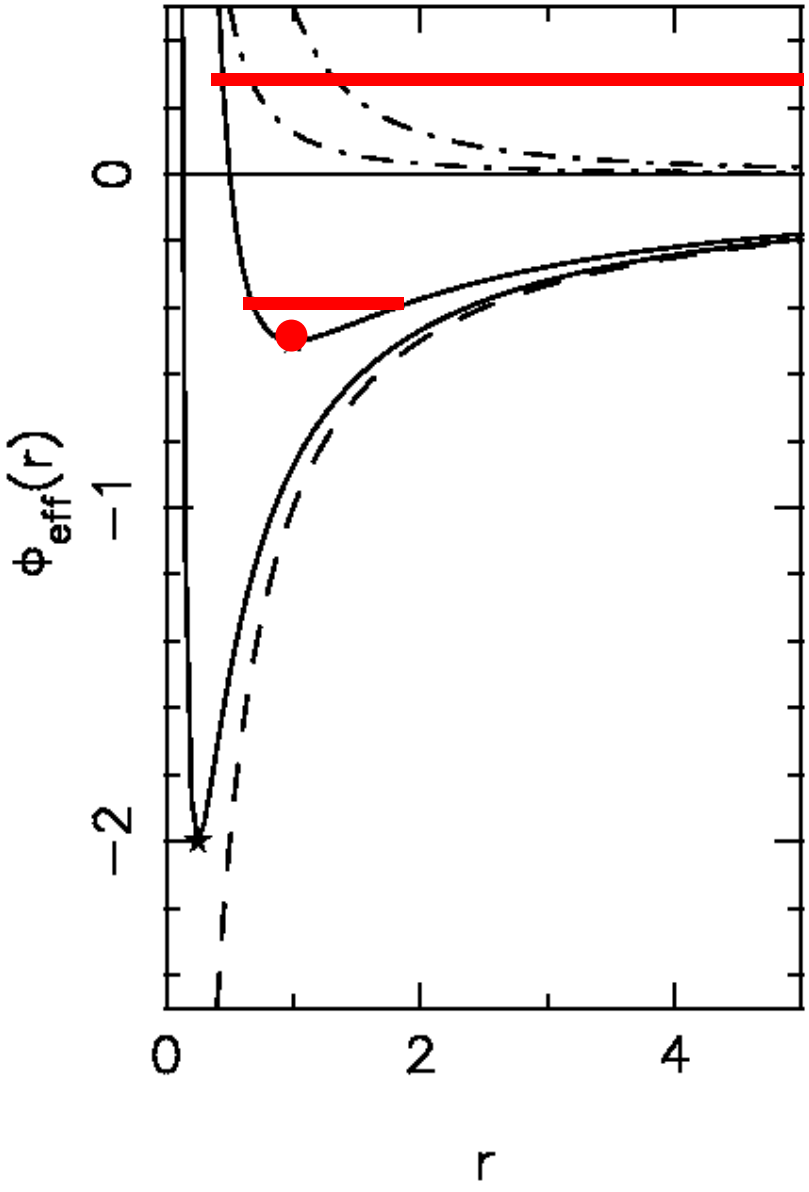
Fix *L*, *E* = min circular

***E* < 0 bound (ellipse)**

***E* > 0 unbound (hyperbola)**

Fix *E* < 0, *L* = max circular

***L* = min radial**



Turning Points

energy and angular momentum :

$$\varepsilon = \frac{\dot{r}^2}{2} + \frac{L^2}{2\,r^2} - \frac{G\,M}{r}$$

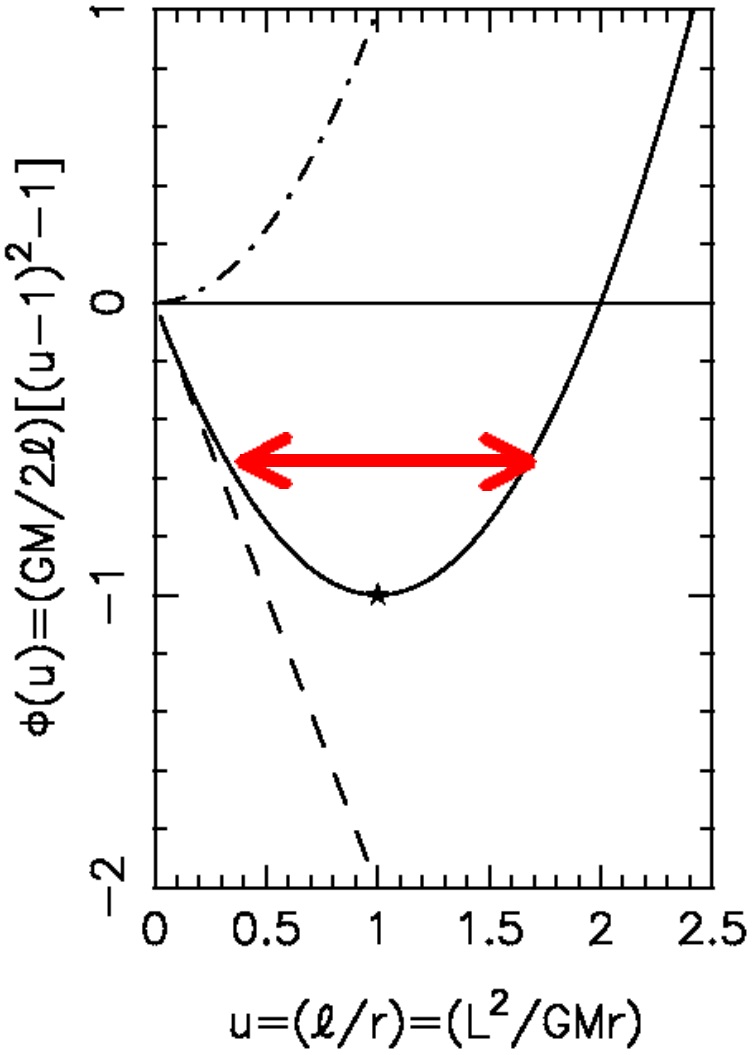
$$= \frac{\dot{r}^2}{2} + \varepsilon_o \left[\left(u - 1 \right)^2 - 1 \right]$$

$$u \equiv \frac{\ell}{r}$$

turning points :

$$\dot{r}^2 = 0 \rightarrow \left(u - 1 \right)^2 = 1 + \frac{\varepsilon}{\varepsilon_o} \equiv e^2$$

$$u^{\pm} = \frac{\ell}{r^{\mp}} = 1 \pm \left(1 + \frac{\varepsilon}{\varepsilon_0} \right)^{1/2}$$



Orbit Shape

$$\varepsilon = \frac{\dot{r}^2}{2} + \varepsilon_0 \left(x^2 - 1 \right)$$

$$\dot{r}^2 = 2\,\varepsilon_0 \left(\frac{\varepsilon}{\varepsilon_0} + 1 - x^2 \right) = \left(\frac{L}{\ell} \right)^2 \left(e^2 - x^2 \right)$$

$$\frac{d\,x}{d\theta} = \frac{\dot{x}}{\dot{\theta}} = \left(-\frac{\ell\,\dot{r}}{r^2} \right) \Big/ \left(\frac{L}{r^2} \right) = -\frac{\ell}{L}\,\dot{r}$$

$$\left(\frac{d\,x}{d\theta} \right)^2 + x^2 = e^2$$

simple harmonic oscillator !

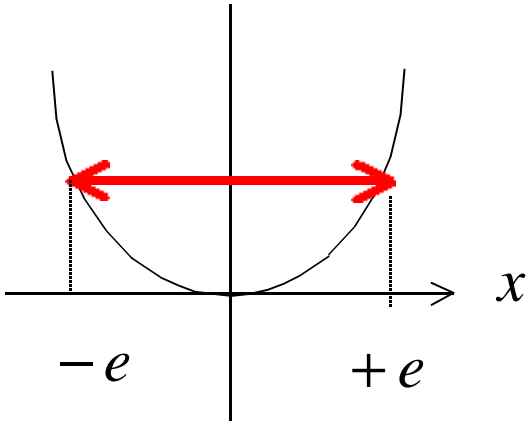
$$x = e\cos\theta \quad \rightarrow \quad \boxed{r = \frac{\ell}{1 + e\cos\theta}}$$

$$x \equiv \frac{\ell}{r} - 1 \qquad \dot{x} = -\frac{\ell\,\dot{r}}{r^2}$$

$$\ell \equiv \frac{L^2}{G\,M} \qquad L \equiv r^2\,\dot{\theta}$$

$$\varepsilon_0 \equiv \frac{1}{2}\left(\frac{L}{\ell} \right)^2$$

$$e^2 \equiv \frac{\varepsilon}{\varepsilon_0} + 1$$



Conic Sections

$$r = \frac{\ell}{1 + e\cos\theta}$$

e = eccentricity

> 1 hyperbola

$= 1$ parabola

< 1 ellipse

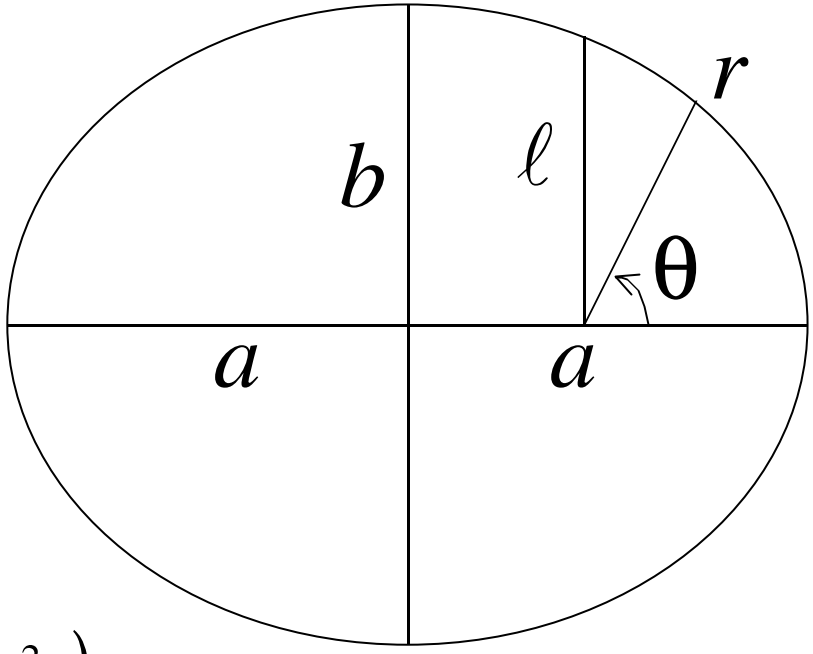
$= 0$ circle

ℓ = semi - latus rectum $= a\left(1 - e^2\right)$

a = semi - major axis

b = semi - minor axis

$= a\left(1 - e^2\right)^{1/2}$



$\theta = 0$	$r = \frac{\ell}{1+e}$	$= a(1-e)$	periastron
$\theta = 90^\circ$	$r = \ell$	$= a(1-e^2)$	
$\theta = 180^\circ$	$r = \frac{\ell}{1-e}$	$= a(1+e)$	apastron

Orbital Speed

$$r = \frac{\ell}{1 + e\cos\theta} \qquad \ell \equiv \frac{L^2}{G\,M} = a\left(1 - e^2\right)$$

$$\varepsilon = -\frac{G\,M}{2\,\ell}\left(1 - e^2\right) = -\frac{G\,M}{2\,a}$$

Energy :

$$V^2 = 2\,\varepsilon + \frac{2\,G\,M}{r} = G\,M\left[\frac{2}{r} - \frac{1}{a}\right]$$

Motion in Time

$$r = \frac{\ell}{1 + e\cos\theta} \qquad \ell \equiv \frac{L^2}{G\,M}$$

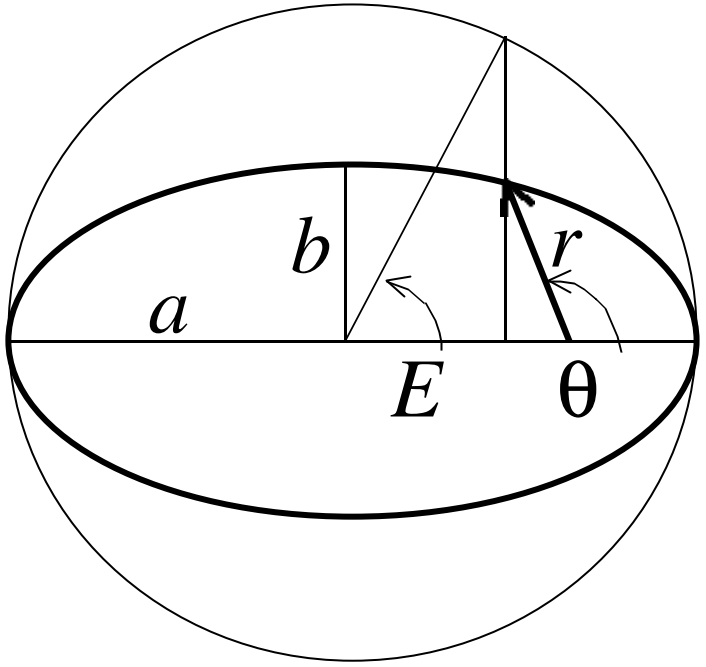
$$\dot{e} = \frac{L}{r^2} = \frac{L}{\ell^2}\,(1 + e\cos\theta)^2$$

$$\frac{d\theta}{(1 + e\cos\theta)^2} = \frac{L}{\ell^2}\,dt$$

$$\int \frac{d\theta}{(1 + e\cos\theta)^2} = \frac{L}{\ell^2}\,(t - T)$$

No analytic solution for $\theta(t)$.

Eccentric Anomaly



$$b = a\sqrt{1-e^2} =$$

θ = true anomaly

E = eccentric anomaly

$$x = a\cos E = a\,e + r\cos\theta$$

$$y = b\sin E = r\sin\theta$$

$$r^2 = a^2\left(\cos^2 E - e\right)^2 + a^2\left(1-e^2\right)\sin^2 E$$

$$r = a\left(1 - e\cos E\right) \rightarrow dr = a\,e\sin E\,dE$$

.....Kepler's equation giving $E(t)$

Motion in Time

η = mean anomaly

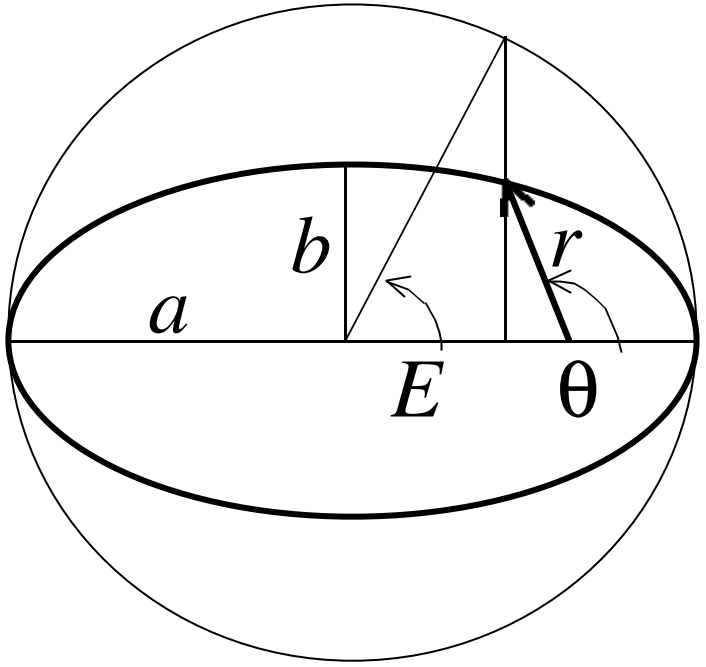
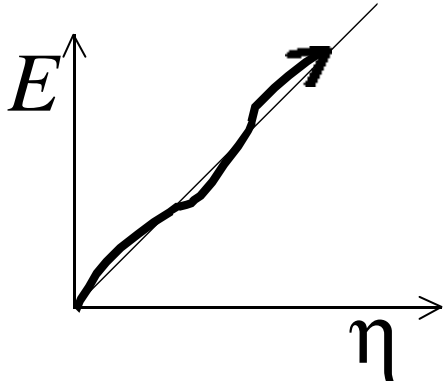
ϕ = orbital phase

P = orbital period

T = time of periastron passage

Kepler's equation :

$$E - e \sin E = \eta = 2\pi\ \phi = \frac{2\pi}{P}(t - T)$$

iterate to find $E(t)$

$$\tan\left(\frac{\theta}{2}\right) = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$$