Radial Potential

energy and angular momentum:

$$\varepsilon \equiv \frac{E}{m} = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{GM}{r}$$

$$L \equiv \frac{J}{m} = r^2\dot{\theta}$$

effective potential: $\varepsilon = \frac{\dot{r}^2}{2} + \Phi(r)$

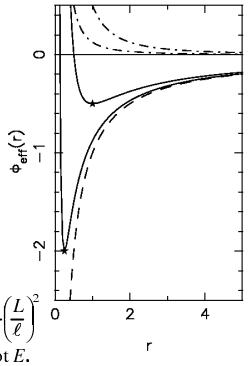
$$\Phi(r) = \frac{L^2}{2r^2} - \frac{GM}{r} = \varepsilon_o\left(\frac{\ell^2}{r^2} - 2\frac{\ell}{r}\right)$$

circular orbit:

$$\frac{\partial\Phi}{\partial r} = \frac{GM}{r^2} - \frac{L^2}{r^3} = 0 \rightarrow r = \frac{L^2}{GM} \equiv \ell$$

$$\Phi(\ell) = -\varepsilon_o \quad \varepsilon_o \equiv \frac{GM}{2\ell} = \frac{1}{2}\left(\frac{GM}{L}\right)^2 = \frac{1}{2}\left(\frac{L}{\ell}\right)^2$$

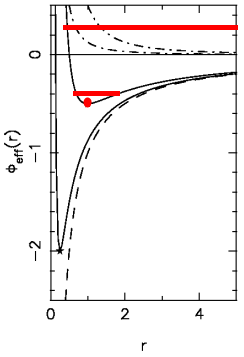
Note: ℓ, ε_o dependon M and J butnot E.



Types of Orbits

Fix L, E = min circular
$E < 0$ bound (ellipse)
$E > 0$ unbound (hyperbola)

Fix $E < 0$, L = max circular
L = min radial



Turning Points

energy and angular momentum :

$$\varepsilon = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} - \frac{GM}{r}$$

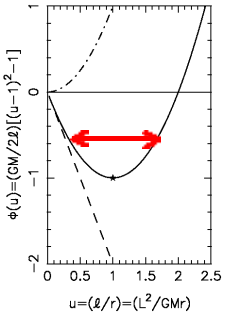
$$= \frac{\dot{r}^2}{2} + \varepsilon_o\left[(u-1)^2 - 1\right]$$

$$u \equiv \frac{\ell}{r}$$

turningpoints:

$$\dot{r}^2 = 0 \rightarrow (u-1)^2 = 1 + \frac{\varepsilon}{\varepsilon_o} \equiv e^2$$

$$u^{\pm} = \frac{\ell}{r^{\pm}} = 1 \pm \left(1 + \frac{\varepsilon}{\varepsilon_0}\right)^{1/2}$$



Orbit Shape

$$\varepsilon = \frac{\dot{r}^2}{2} + \varepsilon_0\left(x^2 - 1\right)$$

$$\dot{r}^2 = 2\varepsilon_0\left(\frac{\varepsilon}{\varepsilon_0} + 1 - x^2\right) = \left(\frac{L}{\ell}\right)^2\left(e^2 - x^2\right)$$

$$\frac{dx}{d\theta} = \frac{\dot{x}}{\dot{\theta}} = \left(-\frac{\ell\dot{r}}{r^2}\right) \Big/ \left(\frac{L}{r^2}\right) = -\frac{\ell}{L}\dot{r}$$

$$\left(\frac{dx}{d\theta}\right)^2 + x^2 = e^2$$

simple harmonic oscillator !

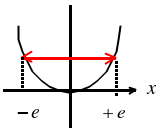
$$x = e\cos\theta \rightarrow \boxed{r = \frac{\ell}{1 + e\cos\theta}}$$

$$x \equiv \frac{\ell}{r} - 1 \quad \dot{x} = -\frac{\ell\dot{r}}{r^2}$$

$$\ell \equiv \frac{L^2}{GM} \quad L \equiv r^2\dot{\theta}$$

$$\varepsilon_0 \equiv \frac{1}{2}\left(\frac{L}{\ell}\right)^2$$

$$e^2 \equiv \frac{\varepsilon}{\varepsilon_0} + 1$$



Conic Sections

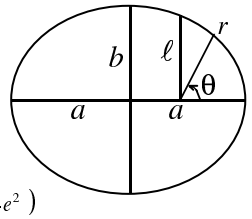
$$r = \frac{\ell}{1 + e\cos\theta}$$

e = eccentricity
- > 1 hyperbola
- $= 1$ parabola
- < 1 ellipse
- $= 0$ circle

ℓ = semi-latus rectum $= a\left(1 - e^2\right)$

a = semi-major axis

b = semi-minor axis $= a\left(1 - e^2\right)^{1/2}$



$\theta = 0 \quad r = \frac{\ell}{1+e} = a(1-e)$ periastron

$\theta = 90^\circ \quad r = \ell = a(1-e^2)$

$\theta = 180^\circ \quad r = \frac{\ell}{1-e} = a(1+e)$ apastron

Orbital Speed

$$r = \frac{\ell}{1 + e\cos\theta} \qquad \ell \equiv \frac{L^2}{GM} = a\left(1 - e^2\right)$$

$$\varepsilon = -\frac{GM}{2\ell}\left(1 - e^2\right) = -\frac{GM}{2a}$$

Energy:

$$V^2 = 2\varepsilon + \frac{2GM}{r} = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

Motion in Time

$$r = \frac{\ell}{1 + e\cos\theta} \qquad \ell \equiv \frac{L^2}{G\,M}$$

$$\dot{e} = \frac{L}{r^2} = \frac{L}{\ell^2}(1 + e\cos\theta)^2$$

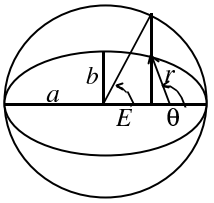
$$\frac{d\theta}{(1 + e\cos\theta)^2} = \frac{L}{\ell^2}dt$$

$$\int \frac{d\theta}{(1 + e\cos\theta)^2} = \frac{L}{\ell^2}(t - T)$$

No analytic solution for $\theta(t)$

Eccentric Anomaly

auxiliary circle



$b = a\sqrt{1 - e^2} =$

Θ = true anomaly

E = eccentric anomaly

$x = a\cos E = a\,e + r\cos\Theta$

$y = b\sin E = r\sin\Theta$

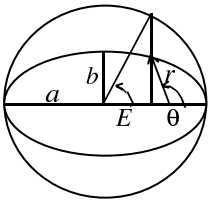
$r^2 = a^2(\cos^2 E - e)^2 + a^2(1 - e^2)\sin^2 E$

$r = a(1 - e\cos E) \rightarrow dr = a\,e\sin E\,dE$

.....Kepler's equation giving $E(t)$

Motion in Time

auxiliary circle



η = mean anomaly

ϕ = orbital phase

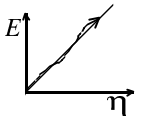
P = orbital period

T = time of periastron passage

Kepler's equation:

$$E - e\sin E = \eta = 2\pi\phi = \frac{2\pi}{P}(t - T)$$

iterate to find $E(t)$



$$\tan\left(\frac{\theta}{2}\right) = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{E}{2}\right)$$