Perturbed Two-Body motion

- **"close" to 2-body motion**
  - planetary systems
  - multi-star systems (triples, quadruples, ...)
  - close binaries (non-spherical stars)
  - general relativity corrections (Mercury, binary pulsars)
- **Perturbed potential**

$$\ddot{O} = \ddot{O}_0 + S$$

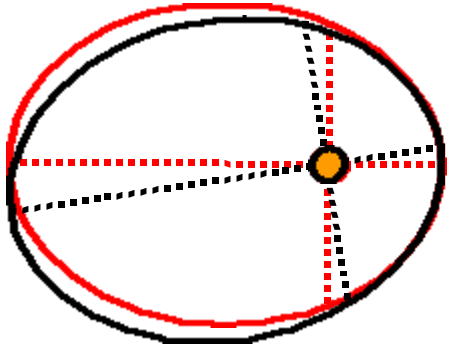
point mass potential

small corrections

the ***disturbing function***

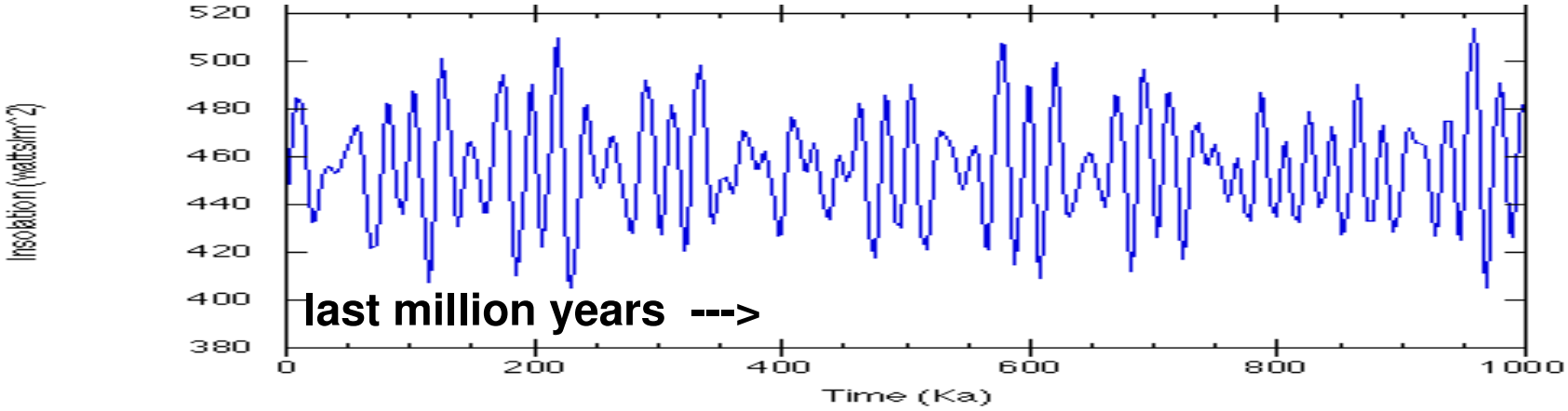
Lagrange's Planetary Equations

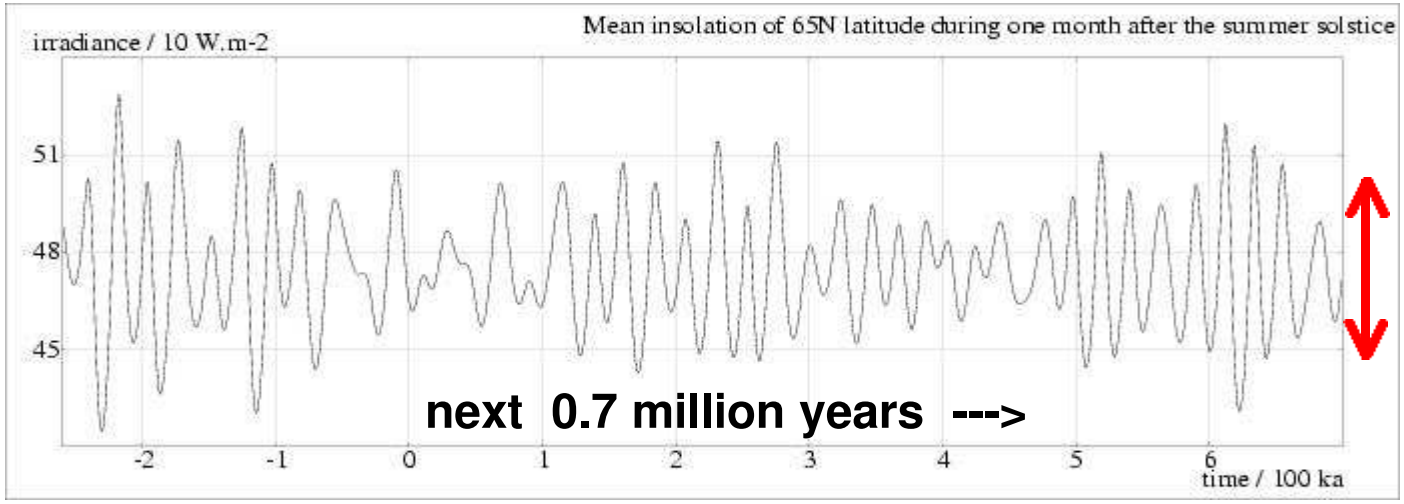
- **Developed for Solar System Planets**
  - main potential due to the Sun
  - secondary potentials from other planets
  - slow <u>secular</u> and/or <u>periodic</u> changes to orbital elements due to orbit-averaged effect of *S*
- **Milankovitch Cycles**
  - effects on Earth's climate
- **Apsidal Motion**
  - precession of the orbit in its own plane
  - e.g. Mercury (43 arcsec / century)
  - tests General Relativity
  - observable in binaries (few deg / year)
  - tests stellar structure theory



Milankovitch Cycles

- **Northern hemisphere insolation, including**
  - **rotation axis precession (23 ka), tilt (41 ka)**
  - **orbit eccentricity (100 ka)**





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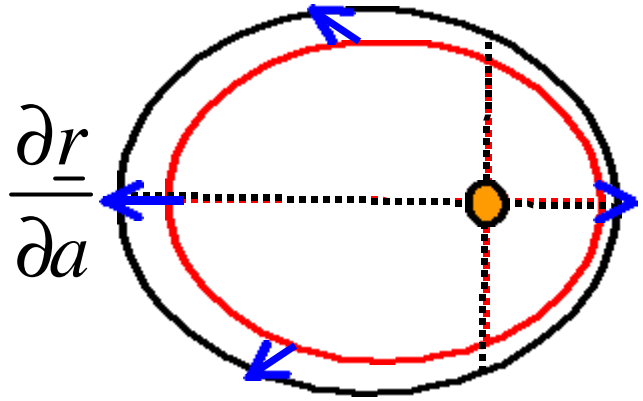
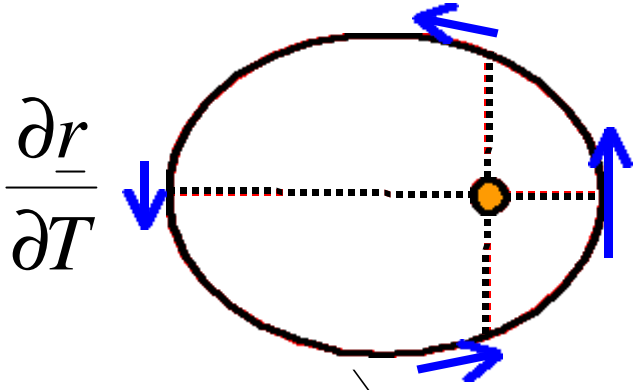
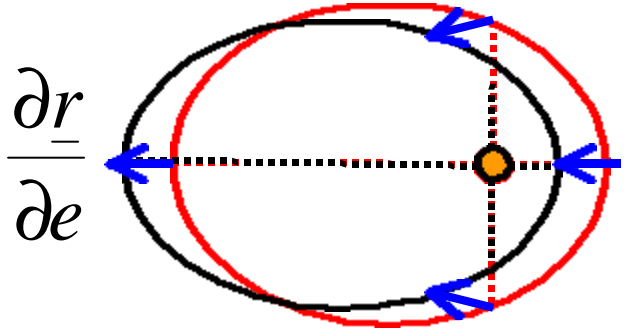
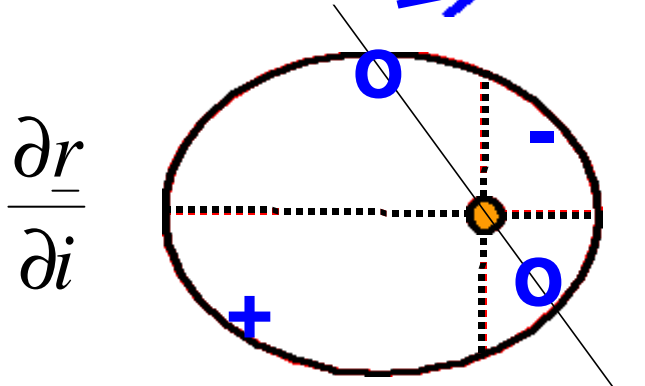
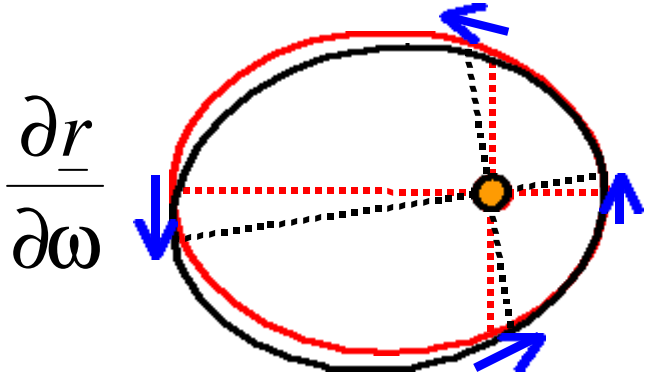
changes in 6 orbital elements

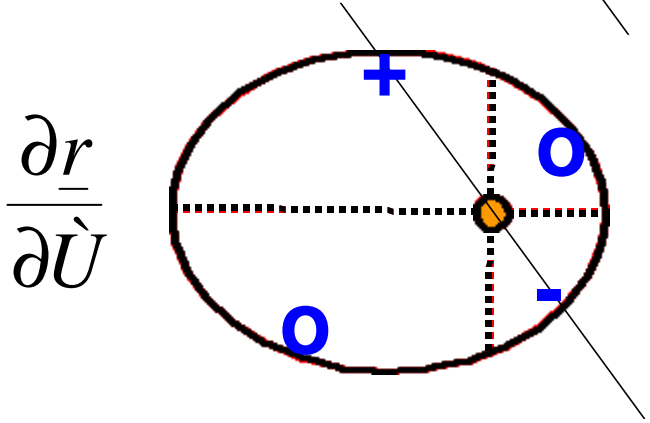












Lagrange's Equations

Orbit depends on time and 6 orbital elements

$\underline{r}(t, a, e, i, \grave{u}, \grave{U}, T)$ $\qquad 4\pi^2 G M P^2 = a^3$

$$\dot{\underline{r}} = \frac{\partial \underline{r}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i} \dot{e}_i \quad \text{where} \quad e_i = (a, e, i, \grave{u}, \grave{U}, T)$$

$$\frac{\partial^2 \underline{r}}{\partial t^2} + \frac{d}{dt}\left[\sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i} \dot{e}_i \right] = -\underline{\nabla}\Phi_0 - \underline{\nabla} S$$

2 - body orbit — slow changes in orbital elements

$$\frac{d}{dt}\left[\sum_{i=1}^{6} \frac{\partial \underline{r}}{\partial e_i} \dot{e}_i \right] \cdot \frac{\partial \underline{r}}{\partial e_k} = -\underline{\nabla} S \cdot \frac{\partial \underline{r}}{\partial e_k}$$

$$\sum_i M_{ik} \dot{e}_i = -\int \frac{\partial S}{\partial e_k} dt \quad \rightarrow \quad \boxed{\dot{e}_i = -\sum_k M^{-1}_{ik} \int \frac{\partial S}{\partial e_k} dt}$$

Lagrange's planetary equations

$$\dot{a} = \frac{2}{n\,a}\frac{\partial S}{\partial \chi}; \qquad \dot{e} = \frac{1}{n\,a^2 e}\left[(1-e^2)\frac{\partial S}{\partial \chi} - \sqrt{1-e^2}\,\frac{\partial S}{\partial \omega} \right]$$

$$\dot{\chi} = -\frac{(1-e^2)}{n\,a^2 e}\frac{\partial S}{\partial e} - \frac{2}{n\,a}\frac{\partial S}{\partial a}; \qquad \dot{U} = \frac{1}{n\,a^2\sqrt{1-e^2}\,\sin i}\frac{\partial S}{\partial i}$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{n\,a^2\,e}\frac{\partial S}{\partial e} - \frac{\cot i}{n\,a^2\sqrt{1-e^2}}\frac{\partial S}{\partial i}$$

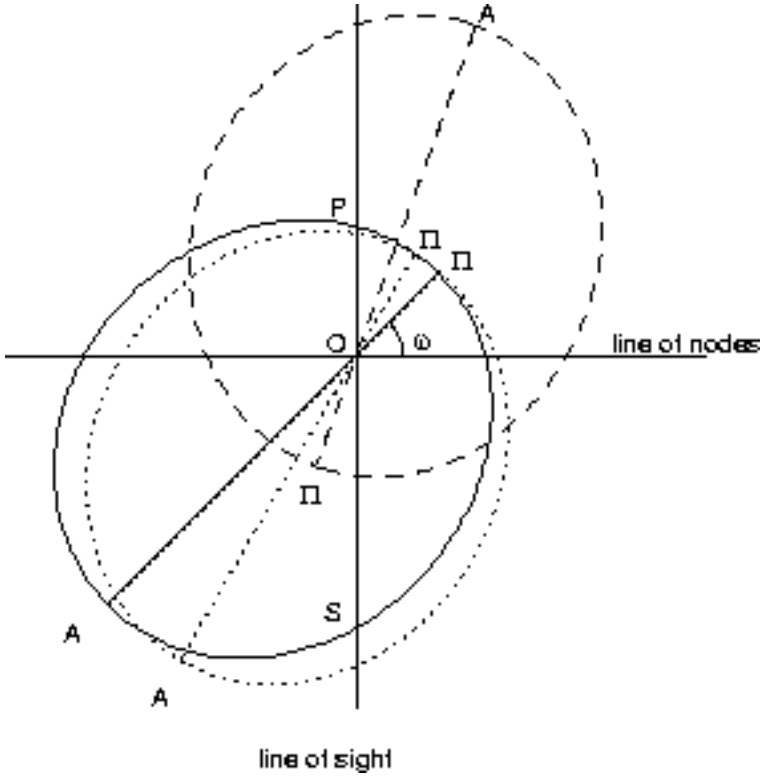
$$\dot{i} = \frac{1}{n\,a^2\sqrt{1-e^2}}\left[\cot i\frac{\partial S}{\partial \omega} - \mathrm{cosec}\, i\frac{\partial S}{\partial \dot{U}} \right]$$

where $n^2 a^3 = G\,M$ and $\chi = -n\,T$,

$n = 2\pi / P$ is the mean daily motion

apsidal motion

– **longitude of periastron, ω, changes as the orbit precesses**



$$\dot{\omega} = \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial S}{\partial e} - \frac{\cot i}{n\,a^2\sqrt{1-e^2}}\frac{\partial S}{\partial i}$$

relativistic apsidal motion

- **Mercury perihelion precession**
- **binary neutron stars (pulsars)**

gravitational potential (Newton + small GR correction)

$$\ddot{O} = -\frac{G\,M}{r}\left[\,1+\left(\frac{V}{c}\right)^2+\ldots\right] \qquad V = L/r$$

$$L^2 = G\,M\;\ell = G\,M\;a\,(1-e^2)$$

disturbing function :

$$S = -\frac{G^2\;M^2\;a\,(1-e^2)}{c^2\;r^3}$$

small inward acceleration

$$-\frac{\partial\,S}{\partial\,r} = -\frac{G^2\;M^2\;a\,(1-e^2)}{c^2\;r^4}$$

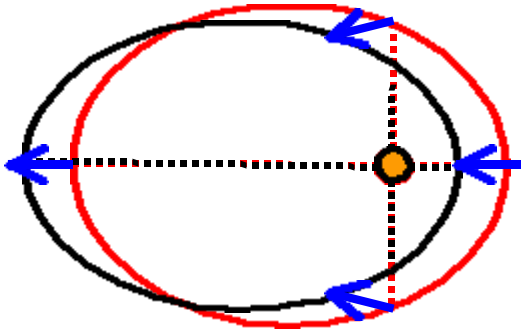
relativistic apsidal motion

$$S = -\frac{G^2 M^2 a(1-e^2)}{c^2 r^3}$$

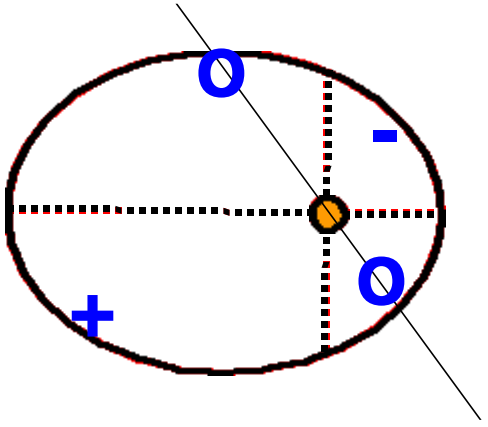
$$\dot{\omega} = \frac{\sqrt{1-e^2}}{n a^2 e}\frac{\partial S}{\partial e} - \frac{\cot i}{n a^2\sqrt{1-e^2}}\frac{\partial S}{\partial i}$$

$\frac{\partial S}{\partial i} \to 0$

$$\frac{\partial S}{\partial e} \equiv \left\langle \nabla S \cdot \frac{\partial \underline{r}}{\partial e} \right\rangle$$



$\frac{\partial \underline{r}}{\partial e}$



$\frac{\partial \underline{r}}{\partial i}$

orbit average of disturbing function :

$$\langle S \rangle = -\left\langle \frac{G^2 M^2 a (1-e^2)}{c^2 r^3} \right\rangle = -\left(\frac{G M}{c a} \right)^2 \frac{1}{(1-e^2)^{1/2}}$$

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{P} \int_0^P \frac{dt}{r^3} = \frac{1}{P} \int_0^{2\pi} \frac{1}{r} \frac{d\theta}{r^2 \dot{\theta}}$$

$$= \frac{1}{P} \int_0^{2\pi} \frac{(1+e\cos\theta)}{\ell} \frac{d\theta}{L}$$

$$= \frac{2\pi}{P \ell L}$$

$$= \frac{1}{a^3 \left(1-e^2\right)^{3/2}}$$

$$r^2 \dot{\theta} = L = \sqrt{G M \ell}$$

$$r = \frac{\ell}{1+e\cos\theta}$$

$$\ell = a\left(1-e^2\right)$$

$$\frac{2\pi}{P L} = \sqrt{\frac{G M}{a^3}} \frac{1}{\sqrt{G M \ell}}$$

$$= \frac{1}{a^2 \sqrt{1-e^2}}$$

An incorrect calculation

apsidal motion :

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{n\,a^2 e}\frac{\partial\, S}{\partial\, e}$$

$$= \frac{G^2\, M^2}{n\, c^2\, a^4 \left(1-e^2\right)}$$

$$= \frac{2\pi\, G\, M}{P\, c^2\, a \left(1-e^2\right)}$$

Factor of 3 too small :{

disturbing function :

$$\langle S \rangle = -\left\langle \frac{G^2\, M^2\, a\,(1-e^2)}{c^2\, r^3} \right\rangle$$

$$= -\left(\frac{G\, M}{c\, a}\right)^2 \frac{1}{(1-e^2)^{1/2}}$$

$$\frac{\partial}{\partial e}\langle S \rangle = \left(\frac{G\, M}{c\, a}\right)^2 \frac{e}{(1-e^2)^{3/2}}$$

wrong because

$$\frac{\partial}{\partial e}\langle S \rangle \neq \left\langle \underline{\nabla} S \cdot \frac{\partial \underline{r}}{\partial e} \right\rangle$$

Ron's derivation

Why this power ?

$$\frac{\partial r}{\partial e} = \frac{\dfrac{\partial}{\partial e}\left\langle r^{-3}\right\rangle}{\dfrac{\partial}{\partial r}\left\langle r^{-3}\right\rangle} = \frac{3\,a^3\,e\,(1-e^2)^{-5/2}}{a^3\,(-3\,r^{-4})}$$

$$\frac{\partial S}{\partial e} = \underline{\nabla} S \cdot \frac{\partial \underline{r}}{\partial e} = \frac{\partial S}{\partial r}\,\frac{\partial r}{\partial e}$$

$$= \frac{-3\,G^2\,M^2\,a\,(1-e^2)}{c^2\,r^4} \times \frac{-e\,r^4}{a^3\,(1-e^2)^{5/2}}$$

$$= \left(\frac{G\,M}{c\,a}\right)^2 \frac{3\,e}{(1-e^2)^{3/2}}$$

- **put into equation for apsidal motion**

Relativistic apsidal motion

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{n\,a^2\,e}\frac{\partial\,S}{\partial\,e}$$

with $$\frac{\partial\,S}{\partial\,e} = \left(\frac{G\,M}{c\,a}\right)^2 \frac{3\,e}{(1-e^2)^{3/2}}$$

and $$n^2 a^3 = G\,M$$

gives $$\boxed{\dot{\omega} = \frac{6\pi\,G\,M}{P\,c^2 a\,(1-e^2)}}$$

Mass determinations

- **combine apsidal motion and mass function to determine the total mass of the system**
- **measure $\delta\omega / \delta t$ over many years**
- **measure $f(m_2)$, and $a_1 \sin(i)$ from pulsar timing**

$$\dot{\omega} = \frac{6\pi \; G \; M}{P \; c^2 a \; (1 - e^2 \;)}$$

  - since $a / M = a_1 / m_2$,

$$\dot{\omega} = \frac{6\pi \; G \; (m_2 \sin i \;)}{P \, c^2 (1 - e^2 \;)(\; a_1 \sin i \;)}$$

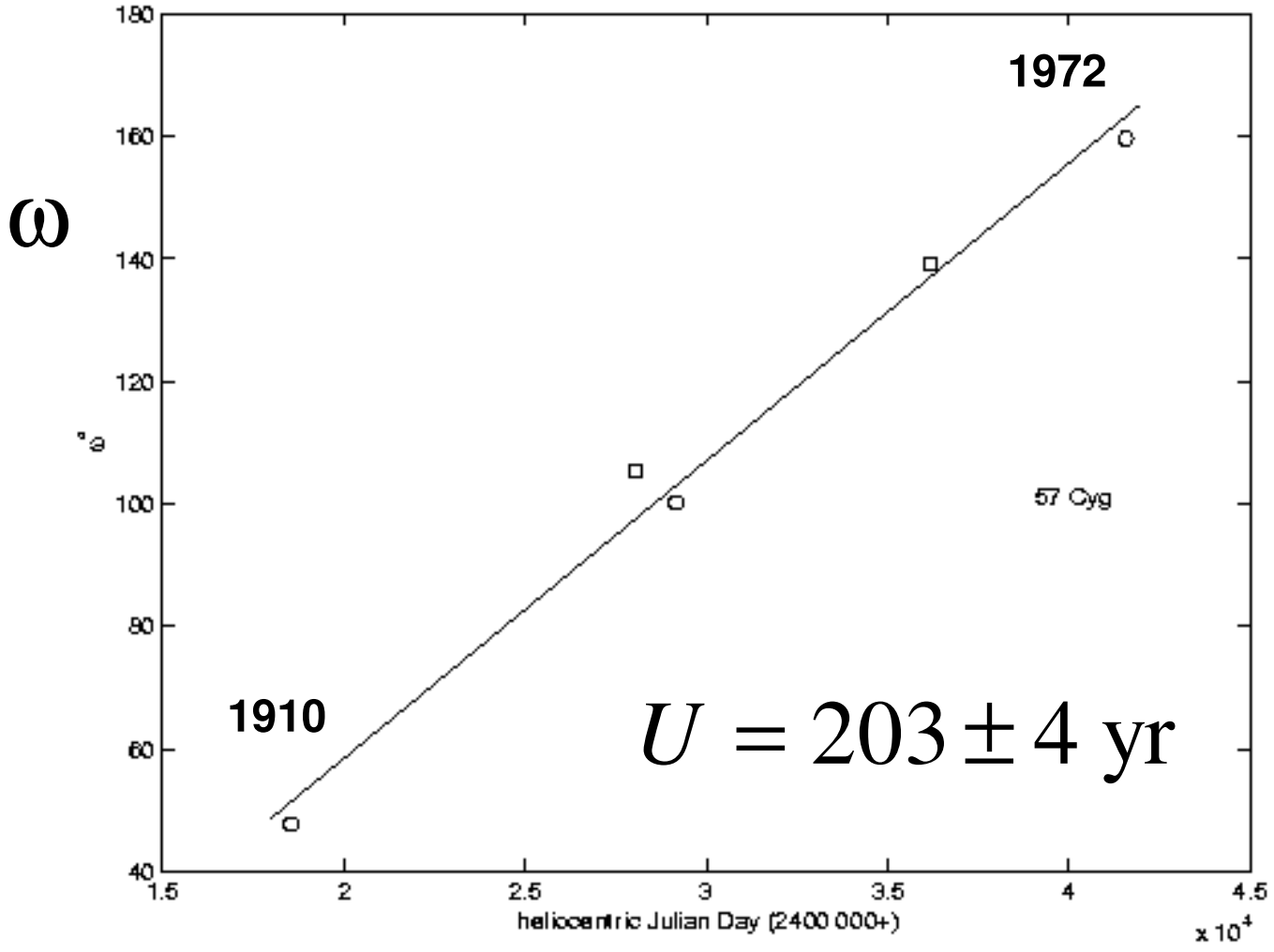
  - solve for $m_2 \sin(i)$, then use $f(m_2) = m_2^3 \sin^3(i) / M^2$

$$\therefore \quad M = \frac{\left(\; m_2 \sin i \; \right)^{3/2}}{\left(\; f(m_2) \; \right)^{1/2}}$$

Aspherical stars and apsidal motion

- **For non-spherical stars**
  - potential not that of 2 point masses
- **observe $\omega(t)$ from radial velocity curves**
- **or from eclipses**
  - changing time of eclipse minima
- **Relate apsidal period to orbital period, *U/P***
  - see handout
  - measureable for *U* ~100s of years
- **tell us about internal structure of stars**
  - need gravitational potential of non-spherical bodies

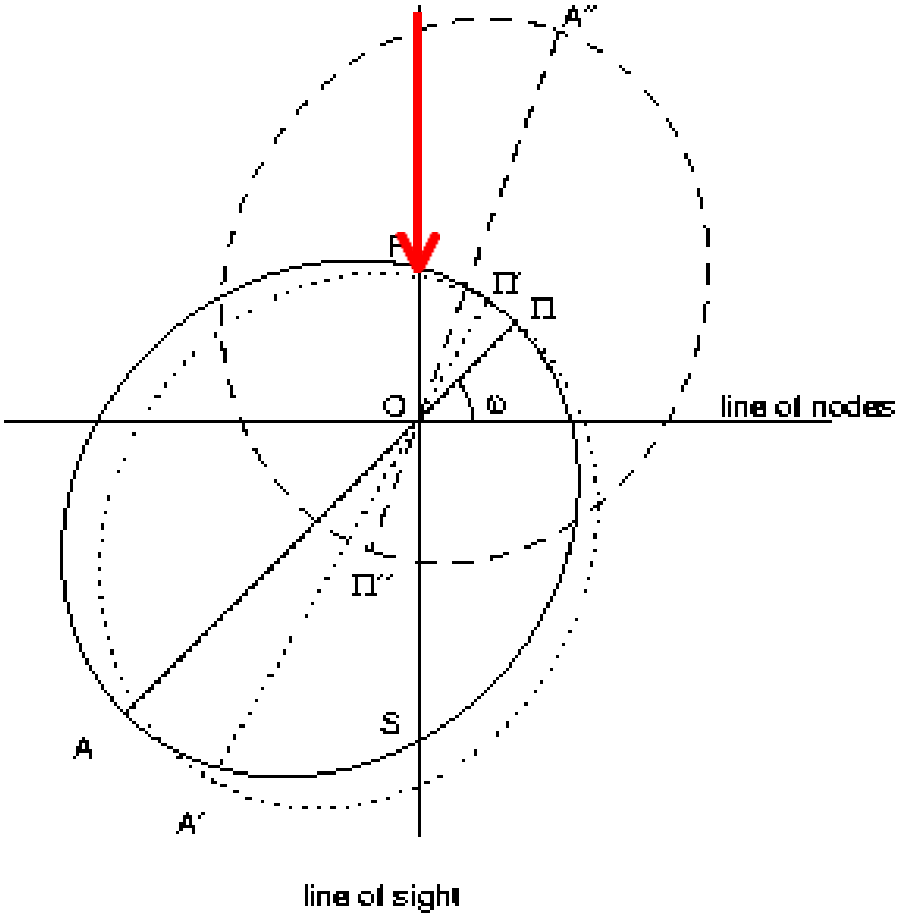
57 Cyg - radial velocity orbit



eclipse times for eccentric orbit

eclipses (for $i = 90^\circ$) at

$\theta + \omega = 90^\circ, 270^\circ$



eclipses at minima of

$$\delta(\theta) = \frac{1-e^2}{1+e\cos\theta}\sqrt{1-\sin^2 i \sin^2(\theta+\omega)}$$

minimum projected separation between stars

eclipse times with apsidal motion

advance of periastron :

$$\omega(t) = \omega_0 + \dot{\omega}\ (\ t - t_0\ )$$

eclipse times :

$$t_n - t_0 = n\ P + \frac{P}{2\pi}\left(\ \omega_0 - \omega_n\ \right) + \frac{e\ P}{\pi}\left(\ \cos\omega_0 - \cos\omega_n\ \right) + \ldots$$

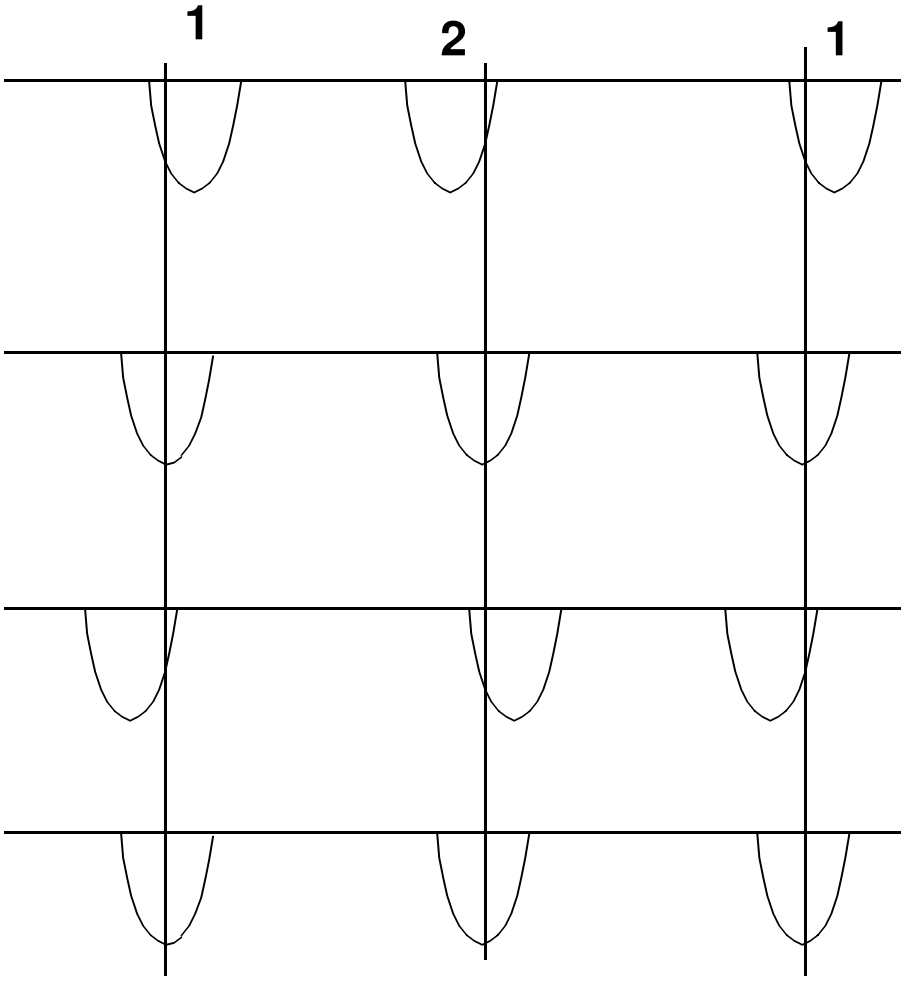
apparent period :

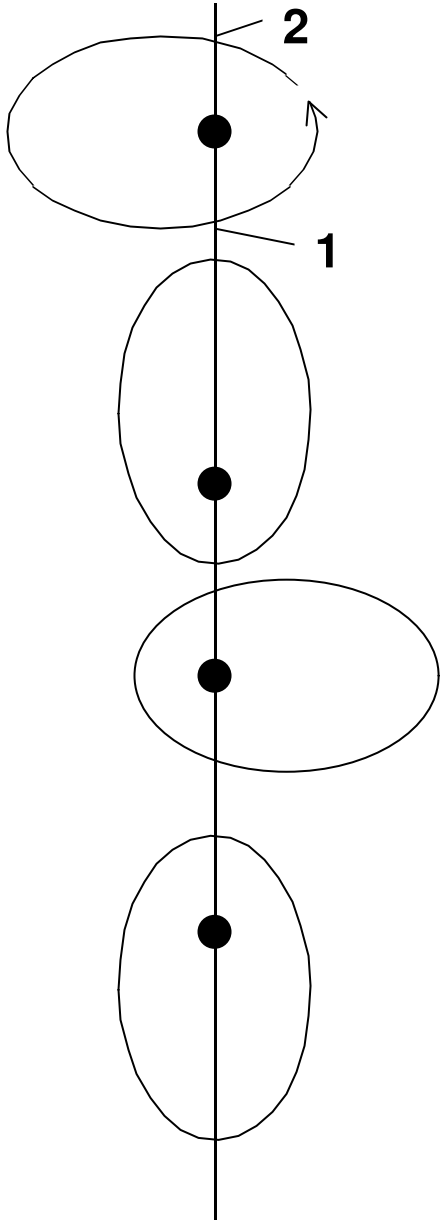
$$P_{sid} \rightarrow P\left(1 - \frac{\dot{\omega}}{2\pi}\right)$$

eclipse phases :

$$\frac{t_n - t_0}{P_{sid}} = n - \frac{e}{\pi}\left(\ \cos\omega - \cos\omega_0\ \right) + ..$$

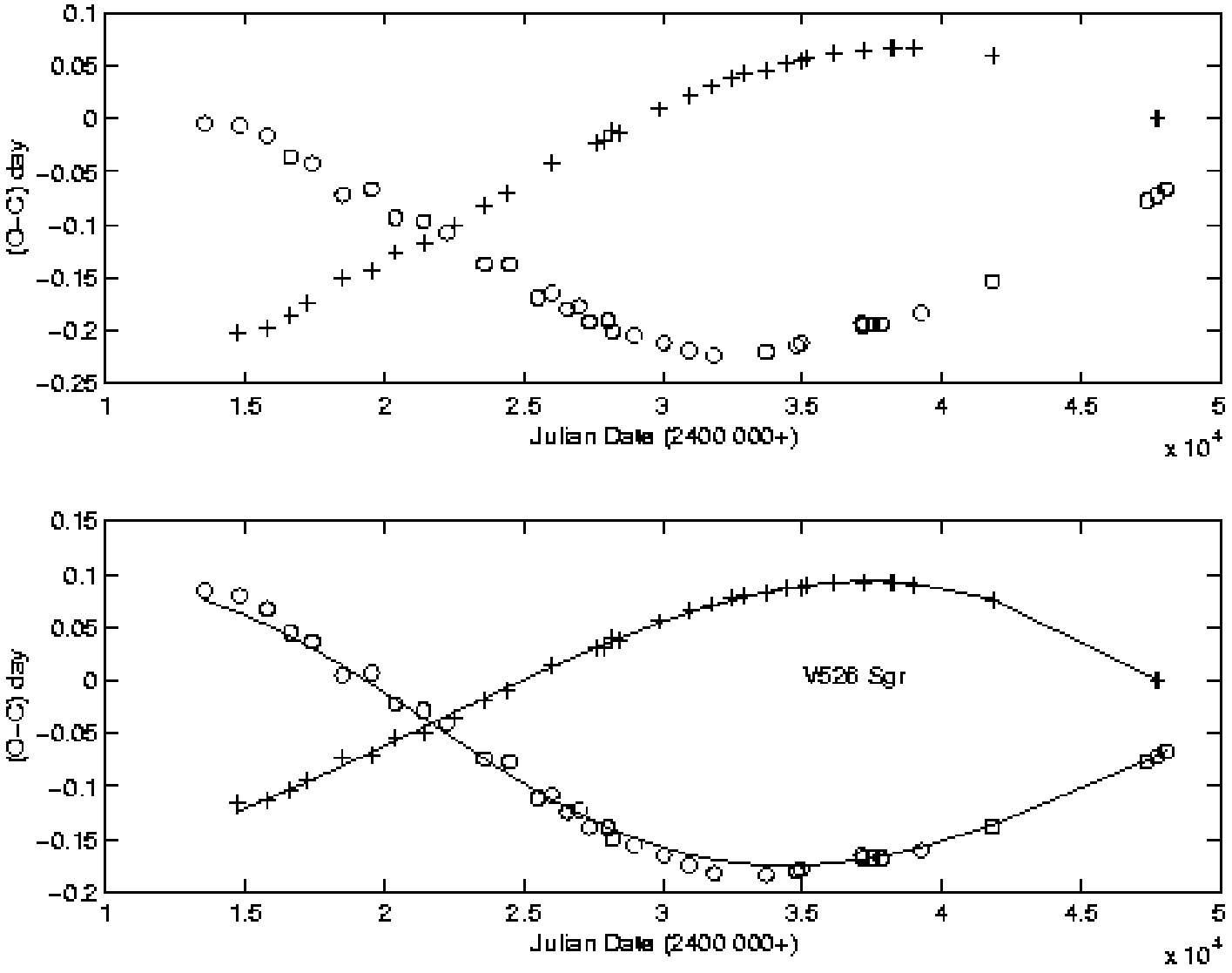
nodding eclipse times





eclipse phases : $\frac{t_n - t_0}{P_{sid}} = n - \frac{e}{\pi}\left(\cos\omega - \cos\omega_0 \right) + ...$

V526 Sgr - eclipse times



Internal Structure of Stars

$$U = \frac{2\pi}{\dot{\omega}} = \text{apsidal} \;\; \text{period}$$

$$\frac{P}{U} = k_1 \left(\frac{R_1}{a} \right)^5 \left(\frac{m_2}{m_1} \left\{ 15 \; f(e) + g(e) \right\} + g(e) \right)$$

$$+ k_2 \left(\frac{R_2}{a} \right)^5 \left(\frac{m_1}{m_2} \left\{ 15 \; f(e) + g(e) \right\} + g(e) \right)$$

$$f(e) = 1 + \frac{13}{2} e^2 + \frac{181}{8} e^4 + \ldots$$

$$g(e) = \left(1 - e^2 \right)^{-2}$$

$R_1, R_2 = \text{radii of stars}$

$k_1, k_2 = \text{apsidal constants -- internal structure -- of stars}$