Lecture 7

Dynamics of the Universe

Solutions to the Friedmann Equation for R(t)

Hubble Parameter Evolution -- H(z)

$$H^2 \equiv \left(\frac{\dot R}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k c^2}{R^2}$$

$$\frac{H^2}{H_0^2} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda - \frac{k c^2}{H_0^2 R_0^2} x^2$$

evaluate at $x = 1 \rightarrow 1 = \Omega_0 - \frac{k c^2}{H_0^2 R_0^2}$

Dimensionless Friedmann Equation:

$$\frac{H^2}{H_0^2} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1 - \Omega_0)\, x^2$$

Curvature Radius today:

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega_0 - 1}} \rightarrow \begin{cases} k = +1 & \Omega_0 > 1 \\ k = 0 & \Omega_0 = 1 \\ k = -1 & \Omega_0 < 1 \end{cases}$$

Density determines Geometry

$$x = 1 + z = R_0 / R$$

$$\rho_c = \frac{3 H_0^2}{8\pi G}$$

$$\Omega_M \equiv \frac{\rho_M}{\rho_c}, \quad \Omega_R \equiv \frac{\rho_R}{\rho_c}$$

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3 H_0^2}$$

$$\Omega_0 \equiv \Omega_M + \Omega_R + \Omega_\Lambda$$

Possible Universes

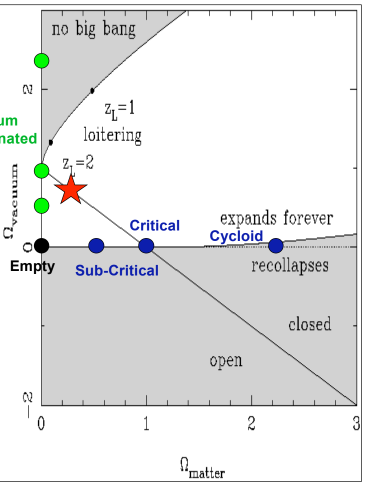
$H_0 \approx 70 \frac{\text{km/s}}{\text{Mpc}}$

$\Omega_M \sim 0.3$

$\Omega_\Lambda \sim 0.7$

$\Omega_R \sim 8\times10^{-5}$

$\Omega = 1.0$



Eternal Static Universe

Einstein introduced Λ

to enable an eternal static universe.

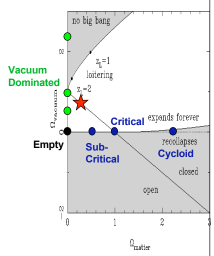
$$\dot R^2 = \left(\frac{8\pi G \rho + \Lambda}{3}\right) R^2 - k c^2$$

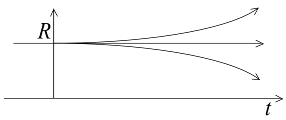
$$\dot R = 0 \rightarrow \Lambda = \frac{3 k c^2}{R^2} - 8\pi G \rho$$

Einstein's biggest blunder. (Or, maybe not.)

Static models unstable.

Fine tuning.





Empty Universe (Milne)

$$\dot R^2 = \left(\frac{8\pi G \rho + \Lambda}{3}\right) R^2 - k c^2$$

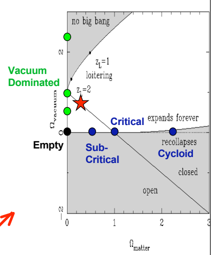
Set $\rho = 0$, $\Lambda = 0$. Then $\dot R^2 = -k c^2$

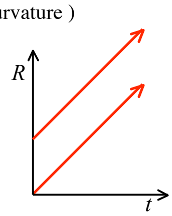
$\rightarrow k = -1$ (negative curvature)

$\dot R = c$, $R = c\, t$

$$H \equiv \frac{\dot R}{R} = \frac{1}{t}$$

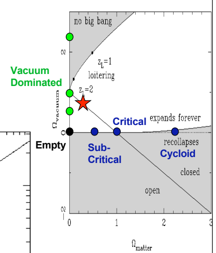
$$\text{age}: \quad t_0 = \frac{R_0}{c} = \frac{1}{H_0}$$

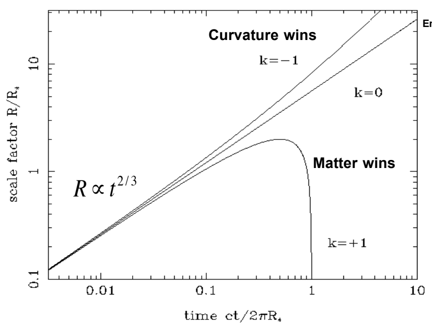




Negative curvature drives rapid expansion/flattening

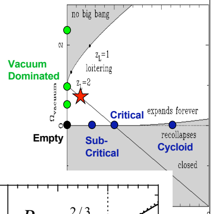
Matter-dominated Universes





All evolve as $R \sim t^{2/3}$ at early times.

Critical Universe
(Einstein - de Sitter)



$$\Omega_M \equiv \frac{\rho_M}{\rho_c} = 1$$

$$\Omega_R = \Omega_\Lambda = 0 \quad \rightarrow \quad k = 0 \ \ (\text{ flat })$$

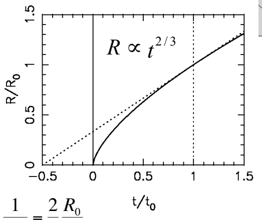
$$\rho = \frac{3 H_0^2}{8\pi G}\left(\frac{R_0}{R}\right)^3$$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 = \frac{H_0^2 R_0^3}{R}$$

$$dR\, R^{1/2} = H_0 R_0^{3/2}\, dt$$

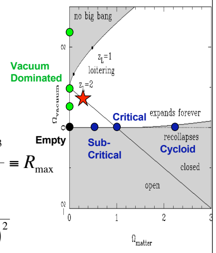
$$\frac{2}{3} R^{3/2} = H_0 R_0^{3/2}\, t$$

$$\frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{2/3}, \qquad \text{age}: \ t_0 = \frac{2}{3}\frac{1}{H_0} = \frac{2}{3}\frac{R_0}{c}$$



Matter decelerates expansion.

Super-Critical Cycloid Universe



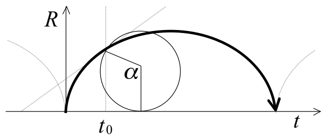
$$\Omega_M > 1, \quad \Omega_R = \Omega_\Lambda = 0 \quad \rightarrow \quad k = +1 \ \ (\text{ closed })$$

$$\dot{R}^2 = \frac{8\pi G \rho_0 R_0^3}{3R} - c^2 \qquad \dot{R} = 0 \rightarrow R = \frac{8\pi G \rho_0 R_0^3}{3c^2} \equiv R_{max}$$

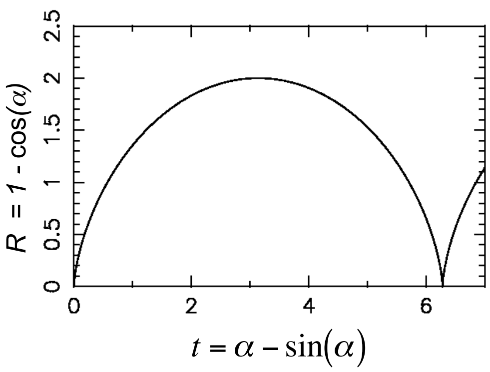
$$R = \frac{R_{max}}{2}(1-\cos\alpha) \qquad H \equiv \frac{\dot{R}}{R} = \frac{2c}{R_{max}}\frac{\sin\alpha}{(1-\cos\alpha)^2}$$

$$t = \frac{R_{max}}{2c}(\alpha - \sin\alpha) \qquad \text{"Big Crunch" at } t = \frac{\pi}{c}R_{max}$$

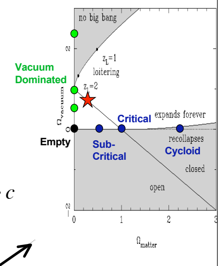
Cycloid curve constructed by rolling a wheel thru angle α.



Cycloid Universe



Sub-Critical Open Universe



$$\Omega_M < 1, \quad \Omega_R = \Omega_\Lambda = 0$$

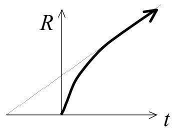
$$\rightarrow \quad k = -1 \ \ (\text{ negative curvature })$$

$$\dot{R}^2 = \frac{8\pi G \rho_0 R_0^3}{3R} + c^2 \qquad R \rightarrow \infty \quad \dot{R} \rightarrow c$$

$$R = R_* \ (\cosh\alpha - 1\ )$$

$$t = \frac{R_*}{c} \ (\sinh\alpha - \alpha\ )$$

$$R_* \equiv \frac{4\pi G \rho_0 R_0^3}{3c^2}$$



Matter initially decelerates, but curvature wins in the end.

Radiation dominated Universe

$$\rho_R = \rho_0 \left(\frac{R_0}{R}\right)^4 \qquad \Omega_M = \Omega_\Lambda = 0$$

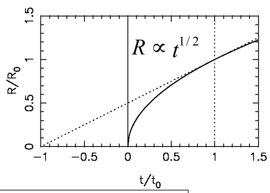
$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - k c^2$$

$$= \frac{8\pi G \rho_0 R_0^4}{3R^2} - k c^2 \approx \frac{H_0^2 R_0^4}{R^2}$$

$$dR\, R = H_0 R_0^2\, dt$$

$$\frac{1}{2}R^2 = H_0 R_0^2\, t$$

$$\frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{1/2}, \qquad \text{age}: \ t_0 = \frac{1}{2 H_0} = \left(\frac{3}{32\pi G \rho_0}\right)^{1/2}$$



Neglect curvature at early times.
Radiation decelerates expansion.

Vacuum dominated Universe

$$\dot{R}^2 = \frac{\Lambda}{3}R^2 - k c^2 \qquad \Omega_M = \Omega_R = 0$$

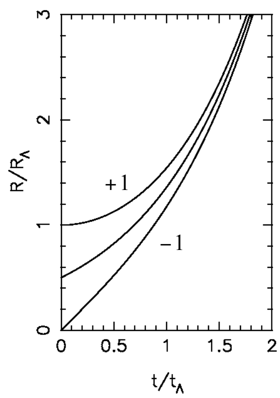
$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3} - \frac{k c^2}{R^2}$$

$$\frac{R}{R_\Lambda} = \begin{cases} \cosh(t/t_\Lambda) & k = +1 \\ \frac{1}{2}\exp(t/t_\Lambda) & k = 0 \\ \sinh(t/t_\Lambda) & k = -1 \end{cases}$$

$$t_\Lambda \equiv \sqrt{3/\Lambda} \qquad R_\Lambda \equiv c\, t_\Lambda$$

$$R_{min} = R_\Lambda \frac{1+k}{2} \qquad H \rightarrow 1/t_\Lambda$$

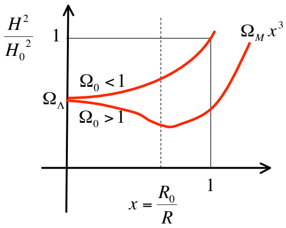
Λ drives exponential expansion
also called inflation



Vacuum Era R(t)

$$\frac{H^2}{H_0^2} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)x^2$$
$$= \Omega_\Lambda + (1-\Omega_0)x^2 + \Omega_M x^3 + \Omega_R x^4$$

$$dt = \frac{-dx}{x\,H(x)} \qquad t = -\int \frac{dx}{x\,H(x)} \qquad x = \frac{R_0}{R}$$

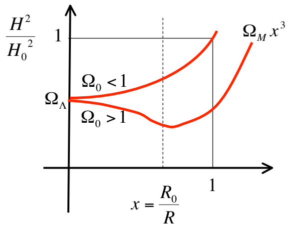


$$H \approx \Omega_\Lambda^{1/2} H_0$$
$$\Omega_\Lambda^{1/2} H_0\, t = -\int \frac{dx}{x}$$
$$= -\ln x = \ln(R/R_0)$$
$$\frac{R}{R_0} = \exp\left(\frac{t}{t_\Lambda}\right) \qquad t_\Lambda = \frac{1}{\Omega_\Lambda^{1/2} H_0}$$

Matter Era R(t)

$$\frac{H^2}{H_0^2} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)x^2$$
$$= \Omega_\Lambda + (1-\Omega_0)x^2 + \Omega_M x^3 + \Omega_R x^4$$

$$dt = \frac{-dx}{x\,H(x)} \qquad t = -\int \frac{dx}{x\,H(x)} \qquad x = \frac{R_0}{R}$$

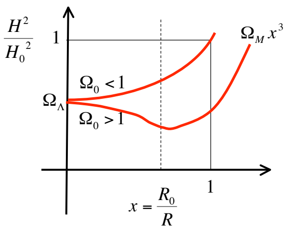


$$H \approx H_0\, \Omega_M^{1/2} x^{3/2}$$
$$\Omega_M^{1/2} H_0\, t = -\int x^{-5/2} dx$$
$$= \frac{2}{3} x^{-3/2} = \frac{2}{3}\left(\frac{R}{R_0}\right)^{3/2}$$
$$\frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{2/3} \qquad t_0 = \frac{2}{3\,\Omega_M^{1/2} H_0}$$

Radiation Era R(t)

$$\frac{H^2}{H_0^2} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)x^2$$
$$= \Omega_\Lambda + (1-\Omega_0)x^2 + \Omega_M x^3 + \Omega_R x^4$$

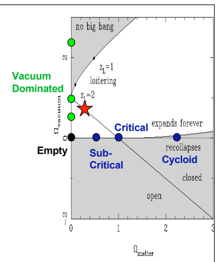
$$dt = \frac{-dx}{x\,H(x)} \qquad t = -\int \frac{dx}{x\,H(x)} \qquad x = \frac{R_0}{R}$$

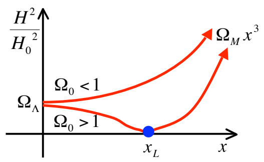


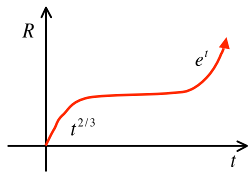
$$H \approx H_0\, \Omega_R^{1/2} x^2$$
$$\Omega_R^{1/2} H_0\, t = -\int x^{-3} dx$$
$$= \frac{1}{2} x^{-2}$$
$$\frac{1}{x} = \frac{R}{R_0} = \left(\frac{t}{t_0}\right)^{1/2} \qquad t_0 = \frac{1}{2\,\Omega_R^{1/2} H_0}$$

Loitering Universes

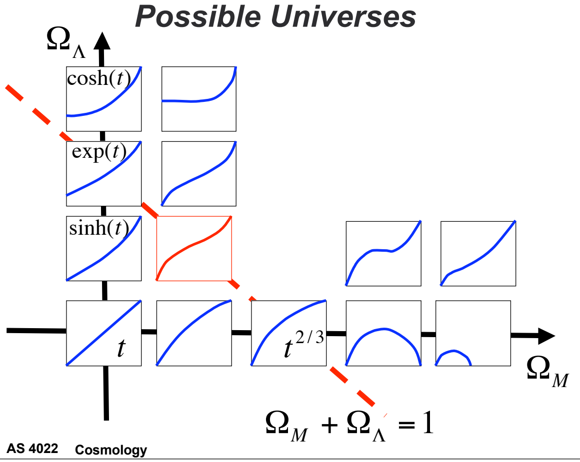
$$\frac{H^2}{H_0^2} = \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)x^2$$
$$\frac{d}{dx}\left(\frac{H^2}{H_0^2}\right) = 3\Omega_M x^2 + 2(1-\Omega_0)x$$
$$\frac{dH^2}{dx} = 0 \quad \Rightarrow x_L = \frac{2(\Omega_0 - 1)}{3\Omega_M}$$







Possible Universes



Our Universe

$$H_0 \approx 70 \,\frac{\text{km/s}}{\text{Mpc}}$$
$$\Omega_M \sim 0.3$$
$$\Omega_\Lambda \sim 0.7$$
$$\Omega_R \sim 8\times 10^{-5}$$
$$\Omega = 1.0$$