7. Analytic Solutions III

- Opacity sources
- Scattering using Eddington Approximation

Sources of Opacity

- Pure hydrogen atmosphere => doesn't look like solar spectrum
- Dominant optical opacity in Sun?
- Dominant optical opacity in A stars?

Negative Hydrogen

- H^- identified as dominant solar opacity
- Only 1 in 10^7 H atoms in solar photosphere is H^-, so why is it so important?
- In optical region, only H atmos in level $n = 3$ can contribute to absorption
- 3646 A < λ < 8206 A => absorption to $n = 3$
- λ > 8206 A => absorption to $n = 4$
- Ionization potential of H^- is 0.754 eV, λ < 1.65μm
- All H^- ions contribute to visual opacity

Compare density of H^- ions with that of neutral hydrogen in the $n = 3$ level at $T = 6000$ K. Use Boltzmann excitation equation

$$\log \frac{n_{0,s}}{n_{0,1}} = \log \frac{g_{0,s}}{g_{0,1}} - \theta \chi_{0,s}$$

$\chi_{0,2}$ = 10.15 eV (Balmer) $g_{0,2} = 2^2$
$\chi_{0,3}$ = 12.10 eV (Paschen) $g_{0,3} = 3^2$

Using Boltzmann gives:

$$\log \frac{n_{0,2}}{n_{0,1}} \cong -7.9 \qquad \frac{n_{0,2}}{n_{0,1}} \cong 1.2 \times 10^{-8}$$

$$\log \frac{n_{0,3}}{n_{0,1}} \cong -9.2 \qquad \frac{n_{0,3}}{n_{0,1}} \cong 6 \times 10^{-10}$$

At T = 6000 K, only about 1 in 10^8 H atoms is not in ground level

Therefore we can approximate total hydrogen number density as

$$\boxed{n_0(H) \approx n_{0,1}(H) \Rightarrow \frac{n_{0,3}(H)}{n_0(H)} = \frac{n_{0,3}(H)}{n_{0,1}(H)}}$$

We now can calculate relative importance of H^- in optical from

$$\boxed{\begin{aligned} \frac{n_{0,3}(H)}{n(H^-)} &= \frac{n_{0,3}(H)}{n_0(H)} \Big/ \frac{n(H^-)}{n_0(H)} \\ \frac{n_{0,3}(H)}{n(H^-)} &= \frac{6\times10^{-10}}{3\times10^{-8}} = 2\times10^{-2} \end{aligned}}$$

Where $n(H^-)/n_0(H)$ calculated before
H^- absorption is about 100 times more than Paschen continuum, ($3646 < \lambda < 8206$) since cross sections are similar. Absorption ~ $n\sigma$

But what about Balmer continuum?

$$\boxed{\frac{n_{0,2}(H)}{n(H^-)} = \frac{1.2\times10^{-8}}{3\times10^{-8}} = 0.4}$$

So in Balmer continuum ($912 < \lambda < 3646$) bound-free opacity due to absorptions to $n = 2$ is comparable to H^- opacity

Opacity is determined by $n\sigma$. σ is from atomic physics and the number density depends on species present and level populations

In solar type stars (T ~ 6000 K), we've shown that in the optical H^- opacity dominates over Paschen continuum opacity (photoionization from $n = 3$), but in the Balmer continuum ($n = 2$) H bound-free is comparable to H^-

How does stellar type effect H^- opacity?

Recall Saha equation using electron pressure

$$\log\frac{N(H^0)P_e}{N(H^-)} = \log\frac{2g_0}{g_-} + 2.5\log T - \theta\chi_r - 1.48$$

$$\log\frac{N(H^-)}{N(H^0)} = \log P_e - \log\frac{2g_0}{g_-} - 2.5\log T + \theta\chi_r - 1.48$$

Depends on P_e, so H^- more important in main sequence stars than giants of same temperature due to P_e proportionality

H and H^- Opacity in A Stars

An A star with T = 10 000 K, log P_e = 2.0, log T = 4.0, θ = 0.5 has

$$\log\frac{N(H^0)}{N(H^-)} = \log\frac{2g_0}{g_-} + 2.5\log T - \theta\chi_r - 1.48 - \log P_e \approx +6.7$$

$$\frac{N(H-)}{N(H^0)} \approx 2\times10^{-7}$$

So $n(H^-)/n_0(H)$ is about six times more than solar value

Compare H atoms in n = 3 (Paschen continuum)

$$\frac{N_{0,3}}{N_{0,1}} \approx 10^{-5} \Rightarrow \frac{N_{0,3}}{N(H^-)} = \frac{10^{-5}}{2\times10^{-7}} \approx 0.5\times10^2$$

About 100 times more H atoms in n = 3 than H^- ions

Neutral H is dominant optical opacity in A stars

Large changes in opacity across Paschen and Balmer limits

Prominent Balmer and Paschen jumps in spectra of A stars

Scattering in Eddington Approximation

- Photons either destroyed or scattered
- Dust scattering & absorption, photon absorbed, heats dust, typically re-radiated at longer wavelength => can treat photon as being destroyed since it doesn't contribute any more at that wavelength
- e^- scattering + hydrogen bound-free
- Resonance line scattering
- Analytic approximation => exam questions…

Absorption & Scattering

Contribution to extinction coefficient due to absorption & scattering:

$$\boxed{\alpha_\nu = \alpha_\nu^a + \alpha_\nu^s}$$

Destruction probability = probability for photon being absorbed

$$\boxed{\varepsilon_\nu = \alpha_\nu^a \big/ (\alpha_\nu^a + \alpha_\nu^s)}$$

Scattering albedo = probability for photon being scattered

$$\boxed{\text{albedo}(\nu) = 1 - \varepsilon_\nu = \alpha_\nu^s \big/ (\alpha_\nu^a + \alpha_\nu^s)}$$

Typical Values

What is numerical value of ε_ν?

Hot star atmospheres: e^- scattering dominates => albedo very high, $\varepsilon_\nu \sim 10^{-4}$. Deep in atmosphere, thermal absorption overwhelms scattering, ε_ν => 1

Cool stars, low metal abundance: Rayleigh scattering off H and H_2 dominates over H^- opacity => albedo large and ε_ν << 1. At large depth H gets excited/ionized, ε_ν => 1

Absorption & Scattering

Contribution to emission coefficient due to absorption:

$$\boxed{j^a = \alpha_\nu^a \, B_\nu}$$

Scattering: photons re-directed into beam, so proportional to mean intensity:

$$\boxed{j^s = \alpha_\nu^a \, J_\nu}$$

Source function for absorption and scattering:

$$\boxed{S_\nu = \frac{j_\nu^a + j_\nu^s}{\alpha_\nu^a + \alpha_\nu^s} = (1 - \varepsilon_\nu)\, J_\nu + \varepsilon_\nu \, B_\nu}$$

Again, use ERT moment equations for isotropic source function:

$$\frac{\mathrm{d}H_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

$$\frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = H_\nu(\tau_\nu)$$

$$\frac{\mathrm{d}^2K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

Apply LTE, source function for scattering + absorption, and Eddington approximation ($J = 3K$) to the "combined moment equation" to get:

$$\frac{1}{3}\frac{\mathrm{d}^2J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = \varepsilon_\nu[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]$$

Also assume Planck function is linear with optical depth:

$$B_\nu(\tau_\nu) = B_{\nu,0} + b\,\tau_\nu \quad \Rightarrow \quad \frac{\mathrm{d}^2B_\nu}{\mathrm{d}\tau_\nu^2} = 0$$

So combined moment equation can be written

$$\frac{1}{3}\frac{\mathrm{d}^2[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]}{\mathrm{d}\tau_\nu^2} = \varepsilon_\nu[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]$$

Which has general solution

$$J_\nu(\tau_\nu) - B_\nu(\tau_\nu) = C_1\,\mathrm{e}^{-\sqrt{3\varepsilon_\nu}\tau_\nu} + C_2\,\mathrm{e}^{+\sqrt{3\varepsilon_\nu}\tau_\nu}$$

Apply boundary conditions to determine constants…

Boundary Conditions: $J_\nu = B_\nu$ for large τ_ν (diffusion approximation)
No incident radiation at surface

First condition gives $C_2 = 0$

To get C_1, could use second Eddington approx, $J_\nu(0) = 2\, H_\nu(0)$, but this is coarse approximation. Therefore set $J_\nu(0) = a_\nu\, H_\nu(0)$, and leave a_ν as a free parameter. This gives:

$$J_\nu(0) = B_{\nu,0} + C_1 = a_\nu\, H_\nu(0) = a_\nu \left[\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu}\right]_{\tau_\nu=0}$$

$$= a_\nu / 3 \left[\frac{\mathrm{d}J_\nu}{\mathrm{d}\tau_\nu}\right]_{\tau_\nu=0} = -(a_\nu / 3)\sqrt{3\varepsilon_\nu}\, C_1 + a_\nu b_\nu / 3$$

Yielding:

$$C_1 = -\frac{B_{\nu,0} - a_\nu b_\nu / 3}{1 + (a_\nu / 3)\sqrt{3\varepsilon_\nu}}$$

So can now determine radiation field:

$$J_\nu(\tau_\nu) = B_\nu(\tau_\nu) + C_1 \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\tau_\nu}$$

$$= B_{\nu,0} + b_\nu \tau_\nu - \frac{B_{\nu,0} - a_\nu b_\nu / 3}{1 + (a_\nu / 3)\sqrt{3\varepsilon_\nu}} \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\tau_\nu}$$

$$S_\nu(\tau_\nu) = B_\nu(\tau_\nu) + (1 - \varepsilon_\nu) C_1 \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\tau_\nu}$$

$$= B_{\nu,0} + b_\nu \tau_\nu - (1 - \varepsilon_\nu) \frac{B_{\nu,0} - a_\nu b_\nu / 3}{1 + (a_\nu / 3)\sqrt{3\varepsilon_\nu}} \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\tau_\nu}$$

$$H_\nu(\tau_\nu) = b_\nu / 3 + \sqrt{\varepsilon_\nu}\, \frac{B_{\nu,0} - a_\nu b_\nu / 3}{\sqrt{3} + a_\nu \sqrt{\varepsilon_\nu}} \mathrm{e}^{-\sqrt{3\varepsilon_\nu}\tau_\nu}$$

$$I_\nu^+(0,\mu) = B_{\nu,0} + b_\nu \tau_\nu - \frac{(1 - \varepsilon_\nu)(B_{\nu,0} - a_\nu b_\nu / 3)}{(1 + (a_\nu / 3)\sqrt{3\varepsilon_\nu})(1 + \mu\sqrt{3\varepsilon_\nu})}$$

Where $I(0,\mu)$ is determined from formal integral solution of ERT

Isothermal Atmosphere with Scattering

In an isothermal atmosphere, there is no τ-dependence on the temperature, and hence the source function is independent of τ, so we set $b_\nu = 0$ giving $B_\nu(\tau_\nu) = B_{\nu,0}$. We also set $a_\nu = 3^{1/2}$ and define:

$$\boxed{\tau_\nu^* = \sqrt{3\varepsilon_\nu}\,\tau_\nu}$$

yielding:

$$\boxed{\begin{aligned} J_\nu(\tau_\nu) &= \left[1 - \frac{1}{1+\sqrt{\varepsilon_\nu}}\,\mathrm{e}^{-\tau_\nu^*}\right] B_{\nu,0} \\ S_\nu(\tau_\nu) &= \left[1 - (1-\sqrt{\varepsilon_\nu})\mathrm{e}^{-\tau_\nu^*}\right] B_{\nu,0} \\ H_\nu(\tau_\nu) &= \frac{1}{\sqrt{3}}\frac{\sqrt{\varepsilon_\nu}}{1+\sqrt{\varepsilon_\nu}}\,\mathrm{e}^{-\tau_\nu^*} B_{\nu,0} \end{aligned}}$$

Surface values, τ = 0, are

$$\boxed{\begin{aligned} J_\nu(0) &= \frac{\sqrt{\varepsilon_\nu}}{1+\sqrt{\varepsilon_\nu}} B_{\nu,0} \\ S_\nu(0) &= \sqrt{\varepsilon_\nu}\, B_{\nu,0} \\ I_\nu^+(0,\mu) &= \frac{1+\mu\sqrt{3}}{1+\mu\sqrt{3\varepsilon_\nu}} B_{\nu,0} \end{aligned}}$$

The form of the source function is known as the “root-epsilon” law and actually holds exactly for isothermal atmosphere, so $a_\nu = 3^{1/2}$ is the correct choice.

Can code this up to test Monte Carlo techniques

Photons at large depth veiled by “fog” of scatterers => can scatter photons to large depth, smaller mean free paths, may be trapped and destroyed.

Thermalization Depth

S_ν departs from B_ν at surface and also deep into atmosphere due to the exp(-τ) factor in S_ν:

$$\boxed{S_\nu(\tau_\nu) = \left[1 - (1 - \sqrt{\varepsilon_\nu})\mathrm{e}^{-\sqrt{3\varepsilon_\nu}\tau_\nu}\right] B_{\nu,0}}$$

S_ν => B_ν only for t > $1/\varepsilon_\nu^{1/2}$. Define ***thermalization depth***:

$$\boxed{\Lambda_\nu = 1/\sqrt{\varepsilon_\nu}}$$

Recall, $\varepsilon_\nu << 1$, so Λ_ν can be very large