11. Spectral Lines II

- Curve of Growth
- Line formation
- Eddington-Milne

Curve of Growth

Curve of Growth is a theoretical plot of line equivalent width against line optical depth, usually plotted as $\log(W_\lambda) - \log(\tau)$.
The $W_\lambda - \tau$ relation is seen in the Schuster-Schwarzschild model atmosphere: continuous spectrum is emitted from a deep layer, $I_c = B_\lambda(T_\mathrm{R})$, second ***reversing layer*** at T_L absorbs the spectral lines.
Radiation we observe is (from L2):

$$\boxed{I_\lambda = B_\lambda(T_\mathrm{R})\,\mathrm{e}^{-\tau_\lambda} + B_\lambda(T_\mathrm{L})\left(1 - \mathrm{e}^{-\tau_\lambda}\right)}$$

where τ_λ is the optical depth of the reversing layer given by:

$$\boxed{\tau_\lambda = \sigma_\lambda\, N_i = \frac{\sqrt{\pi}\,\mathrm{e}^2}{m_\mathrm{e}\, c}\,\frac{\lambda_0^2}{c}\,\frac{f}{\Delta\lambda_D}\, N_i\, H(a,v) \approx \tau_{\lambda_0} H(a,v)}$$

where N_i is the column density along LOS (m^{-2}) of particles in lower level i.

The relative line depression is:

$$\boxed{D_\lambda \equiv \frac{I_c - I_\lambda^l}{I_c} = \frac{B_\lambda(T_\mathrm{R}) - B_\lambda(T_\mathrm{L})}{B_\lambda(T_\mathrm{R})}(1-\mathrm{e}^{-\tau_\lambda}) = D_{\max}\,(1-\mathrm{e}^{-\tau_\lambda})}$$

Where $D_{\max} = [B_\lambda(T_\mathrm{R}) - B_\lambda(T_\mathrm{L})] / B_\lambda(T_\mathrm{R})$ the maximum depression for very strong lines. The equivalent width is

$$\boxed{W_\lambda = D_{\max} \int_{\text{line}} (1-\mathrm{e}^{-\tau_\lambda})\,\mathrm{d}\lambda}$$

showing how W_λ relates to τ and hence also N_i and f.

The curve of growth can be split into three regimes depending on the optical depth (line strength): weak, saturated, and strong lines.

1. Weak Lines: For $\tau_\lambda << 1$, $\exp(-\tau_\lambda) \sim 1 - \tau_\lambda$, so $D \sim D_{\max}\,\tau_\lambda$. The Voigt profile is approximated by Doppler because opacity is too small to map damping wings into emergent spectrum. Replacing $H(a,v)$ with $\exp(-[\Delta\lambda/\Delta\lambda_D]^2)$ with area $\pi^{1/2}\,\Delta\lambda_D$ gives:

$$\boxed{\begin{aligned} D_\lambda &\approx D_{\max}\,\tau_{\lambda_0}\,\mathrm{e}^{-(\Delta\lambda/\Delta\lambda_D)^2} \\ W_\lambda &\approx D_{\max}\,\tau_{\lambda_0}\,\sqrt{\pi}\,\Delta\lambda_D = \frac{\sqrt{\pi}\,\mathrm{e}^2}{m_\mathrm{e}\,c}\frac{\lambda_0^2}{c}\,f\,D_{\max}\,N_i \end{aligned}}$$

$W_\lambda \sim f\,N_i$. Part 1 of curve of growth is ***Doppler part*** and has slope 1:1. Linear increase due to optical thinness of reversing layer.

2. Saturated Lines: For $\tau_\lambda > 1$, line cannot be deeper than $D_{\max}$. The line width increases with τ_{λ_0} and

$$\boxed{W_\lambda \approx Q\,D_{\max}\,\Delta\lambda_D}$$

Where $Q \sim 2 - 4$. This is part 2 of curve of growth or the ***shoulder***.

3. Strong Lines: For $\tau_\lambda >> 1$, the core doesn't change any more. Line centre is fixed at D_{max}, but line wings have $\tau_\lambda < 1$ and may grow in optically thin fashion to increase W_λ. For large τ_{λ_0} the wings contribute a lot because they map the damping part of

$$\boxed{H(a,v) \approx a / \left(\sqrt{\pi}\, v^2\right) = a/\sqrt{\pi}\,(\Delta\lambda_D^2 / \Delta\lambda^2) \sim 1/\Delta\lambda^2}$$

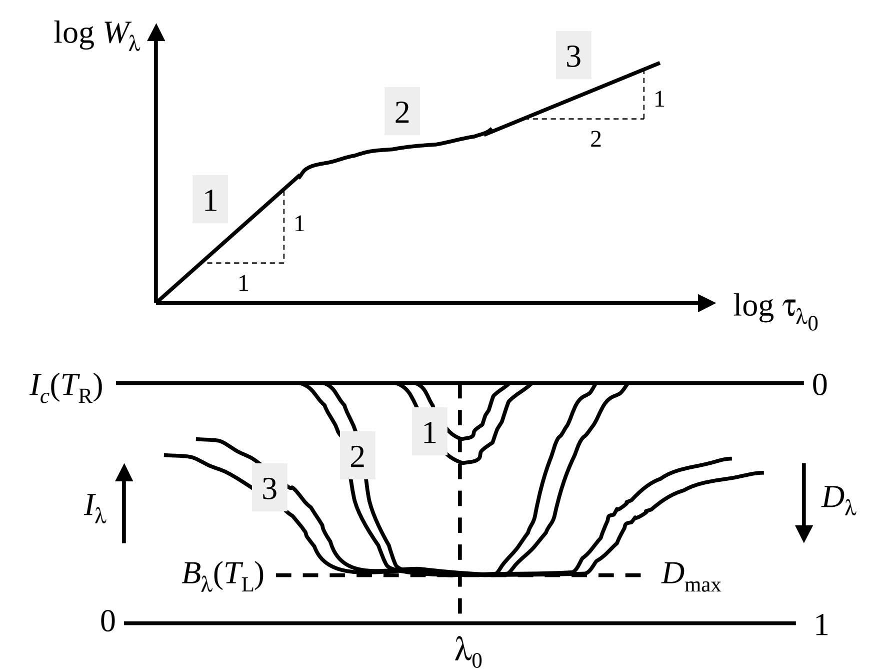
A drop off with $\Delta\lambda$ that is less steep than the exponential decay of the Doppler core. In the damping part of $H(a,v)$ we write:

$$\boxed{\tau_\lambda \approx \tau_{\lambda_0} \frac{a}{\sqrt{\pi}\, v^2} = \tau_{\lambda_0} \frac{a}{\sqrt{\pi}} \frac{\Delta\lambda^2}{\Delta\lambda_D^2}}$$

which gives:

$$\boxed{\begin{aligned} W_\lambda &= D_{\max} \int_{\text{line}} (1 - \mathrm{e}^{-\tau_\lambda})\, \mathrm{d}\lambda \\ &= D_{\max}\, \Delta\lambda_D \sqrt{\tau_{\lambda_0} (a/\sqrt{\pi})} \int_{\text{line}} (1 - \mathrm{e}^{-1/u^2})\, \mathrm{d}u \\ &\sim D_{\max}\, \Delta\lambda_D \sqrt{\tau_{\lambda_0} a} \end{aligned}}$$

Thus part 3 or the ***damping part*** scales as $W_\lambda \sim (a\, \tau_{\lambda_0})^{½} \sim (f\, N_i\, \gamma)^{½}$.



Classical Curve of Growth Fitting

The strategy is to analyze lines via a curve of growth to determine abundances, damping parameter a, excitation temperature T_{exc}, and microturbulence ξ_{micro}. Plot measured W_λs as $\log(W_\lambda/\lambda)$ against

$$\boxed{\log X = \log C + \log(g\ f\ \lambda_0) - \chi\, 5040 / T_{exc}}$$

Where C contains unknowns such as D_{max}, microturbulence, Saha population factor, continuous extinction, elemental abundance.

Changing parameters changes the shape of theoretical curve of growth. By minimizing scatter of data can derive physical parameters.

Residual Flux in a Line

Opacity due to line and continuum: $\boxed{\kappa_\nu = \kappa_\nu^c + \kappa_\nu^l = \kappa^c + \kappa_\nu^l}$

Continuum opacity varies slowly with ν => constant across line.

Write: $\boxed{\eta_\nu = \kappa_\nu^l / \kappa^c}$ and assume η_ν independent of τ.

Now consider total / continuum optical depth:

$$\boxed{\begin{aligned} \mathrm{d}\tau_\nu &= -\rho\,(\kappa^c + \kappa_\nu^l)\,\mathrm{dz} \\ \mathrm{d}\tau &= -\rho\,\kappa^c\,\mathrm{dz} \end{aligned}} \qquad \boxed{\tau_\nu = (1+\eta_\nu)\tau}$$

In ERT want energy created and energy destroyed along beam:

$$\boxed{D = \varepsilon^c \rho\, \kappa^c I_\nu + (1-\varepsilon^c)\rho\, \kappa^c I_\nu + \varepsilon^l \rho\, \kappa^l_\nu I_\nu + (1-\varepsilon^l)\rho\, \kappa^l_\nu I_\nu}$$

continuum absorption	continuum scattering	line absorption	line scattering

$$\boxed{C = \varepsilon^c \rho\, \kappa^c B_\nu + \varepsilon^l \rho\, \kappa^l_\nu B_\nu + (1-\varepsilon^c)\rho\, \kappa^c J_\nu + (1-\varepsilon^l)\rho\, \kappa^l_\nu J_\nu}$$

continuum emission	line emission	continuum scattering	line scattering

ERT becomes:

$$\boxed{\begin{aligned}\frac{\mu}{\rho}\frac{\mathrm{d}I_\nu(\mathrm{z})}{\mathrm{dz}} &= (\varepsilon^c \kappa^c + \varepsilon^l_\nu \kappa^l_\nu)\, B_\nu(\mathrm{z}) \\ &\quad + [(1-\varepsilon^c)\,\kappa^c + (1-\varepsilon^l_\nu)\,\kappa^l_\nu]\, J_\nu(\mathrm{z}) - (\kappa^c + \kappa^l_\nu) I_\nu(\mathrm{z})\end{aligned}}$$

Which simplifies to

$$\boxed{\mu \frac{\mathrm{d}I_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = I_\nu(\tau_\nu) - J_\nu(\tau_\nu) + \lambda_\nu [J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]}$$

where

$$\boxed{\lambda_\nu = \frac{\varepsilon^c \kappa^c + \varepsilon^l_\nu \kappa^l_\nu}{\kappa^c + \kappa^l_\nu} = \frac{\varepsilon^c + \varepsilon^l_\nu \eta_\nu}{1 + \eta_\nu}}$$

Eddington-Milne Solution

Assumptions to solve ERT above:

1. λ_ν independent of τ_ν (i.e., η_ν independent of τ)
2. $B_\nu(T[\tau])$ linear function of τ, continuum optical depth

   $B_\nu = a + b\,\tau = a + p_\nu\,\tau_\nu$

   $p_\nu = b\,/\,(1 + \eta_\nu)$

Can now solve ERT above in similar way to scattering in Eddington solution. Form moment equations:

$$\frac{\mathrm{d}H_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = \lambda_\nu\,[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]$$

$$\frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = H_\nu(\tau_\nu)$$

Use Eddington approximations $J = 3K$, to get from second moment equation above:

$$\frac{\mathrm{d}J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = 3H_\nu(\tau_\nu)$$

Substitute this in first moment equation above and use linearity of B_ν to get:

$$\frac{\mathrm{d}^2 J_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = 3\lambda_\nu\,[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]$$

$$\frac{\mathrm{d}^2[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]}{\mathrm{d}\tau_\nu^2} = 3\lambda_\nu\,[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]$$

Solution: $$J_\nu(\tau_\nu) - B_\nu(\tau_\nu) = C_1\,\mathrm{e}^{-\sqrt{3\lambda_\nu}\,\tau_\nu} + C_2\,\mathrm{e}^{+\sqrt{3\lambda_\nu}\,\tau_\nu}$$

Apply boundary conditions as before: no incident radiation, at large depth $J \Rightarrow B$, Eddington approximation, $J(0)=2H(0)$ or from exact solution $J(0)=3^{1/2}H(0)$

Using these approximations and boundary conditions gives

$$J_\nu = a + p_\nu \tau_\nu + \left(p_\nu - \sqrt{3}\,a\right) \mathrm{e}^{-\sqrt{3\lambda_\nu}\tau_\nu} / \left[\sqrt{3} + \sqrt{3\lambda_\nu}\right]$$

and the emergent flux:

$$H_\nu(0) = J_\nu(0)/\sqrt{3} = \frac{1}{3}\left[p_\nu + \sqrt{3\lambda_\nu}\; a\right] / \left[1 + \sqrt{\lambda_\nu}\right]$$

For the continuum $\eta_\nu = 0$ so we get the residual flux as

$$R_\nu = H_\nu(0)/H^c(0) = \left[\frac{p_\nu + \sqrt{3\lambda_\nu}\; a}{1 + \sqrt{\lambda_\nu}}\right]\left[\frac{1 + \sqrt{\varepsilon^c}}{b + \sqrt{3\varepsilon^c}\, a}\right]$$

Tutorial Example

The above solution has been outlined. Derive it in detail, going through all the steps and algebra. Then derive equations for the residual flux for the following situations

1. Scattering lines: no scattering in continuum, pure scattering in the line
2. Absorption lines: Pure scattering in continuum, no scattering in line

Hint: what are the line and continuum ε values for the above cases?

What are the residual fluxes for very strong lines in the above cases? Comment on the spectral appearance of the line cores.
For a very strong line, consider the limit of very large η_ν