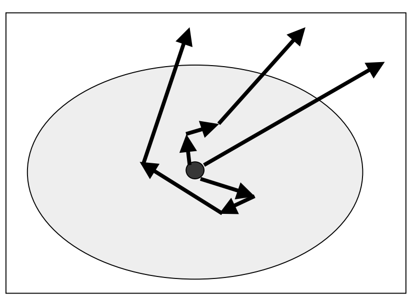
12. Circumstellar Matter
Monte Carlo Radiation Transfer I

- Monte Carlo “Photons” and interactions
- Sampling from probability distributions
- Optical depths, isotropic emission, scattering

Monte Carlo Basics

- Emit energy packet, hereafter a “photon”
- Photon travels some distance
- Something happens…



- Scattering, absorption, re-emission

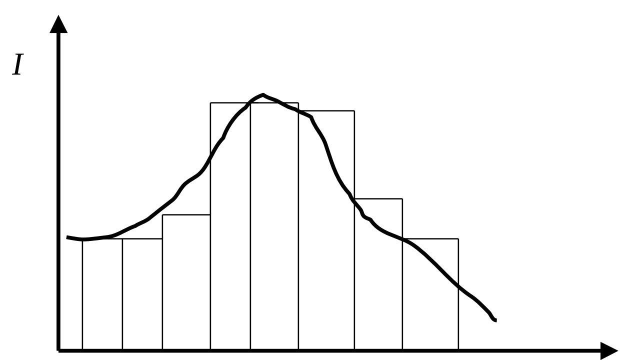
Photon Packets

If the total input luminosity is L, then each photon packet carries energy $E_i = L\,\Delta t\,/\,N$, where N is the number of Monte Carlo photons. A Monte Carlo photon represents N_γ real photons, where $N_\gamma = E_i\,/\,h\nu_i$. So, a Monte Carlo photon packet moving along direction specified by θ will contribute to the specific intensity:

$$\boxed{\begin{aligned} I_\nu &= \frac{\mathrm{d}E_\nu}{\cos\theta\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega} \\ \Delta I_\nu &= \frac{E_i}{\cos\theta\,\Delta A\,\Delta t\,\Delta\nu\,\Delta\Omega} \end{aligned}}$$

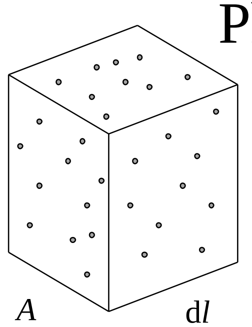
Energy packet

Note, I_ν is a ***distribution function***. We will be working with ***discrete*** energies. Binning the photon packets into directions, frequencies, etc, enables us to simulate a distribution function. e.g., spectrum: bin in frequency; scattering phase function: bin in angle



ν (spectrum)
θ (phase function)

Photon Interactions



Volume = $A\,\mathrm{d}l$

Number density n

Cross section σ

Energy removed from beam per t / ν / $\mathrm{d}\Omega = I_\nu\,\sigma$

Number of photons absorbed/scattered from beam per sec

$$= I_\nu\,\sigma\,n\,A\,\mathrm{d}l$$

Number of photons absorbed/scattered from beam per sec per area

$$= I_\nu\,\sigma\,n\,\mathrm{d}l$$

Intensity differential over $\mathrm{d}l$ is $\mathrm{d}I_\nu = -\,I_\nu\,n\,\sigma\,\mathrm{d}l$. Therefore

$$I_\nu\,(l) = I_\nu\,(0)\,\exp(-n\,\sigma\,l)$$

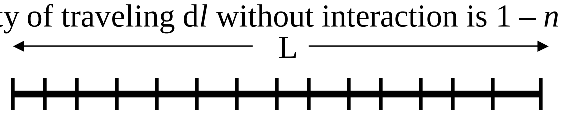
Fraction scattered or absorbed / length = $n\,\sigma$

$n\,\sigma$ = volume absorption coefficient = $\rho\,\kappa$

Mean free path = 1 / n σ = average distance between interactions

Probability of interaction over $\mathrm{d}l$ is $n\,\sigma\,\mathrm{d}l$

Probability of traveling $\mathrm{d}l$ without interaction is $1 - n\,\sigma\,\mathrm{d}l$



N segments of length L / N

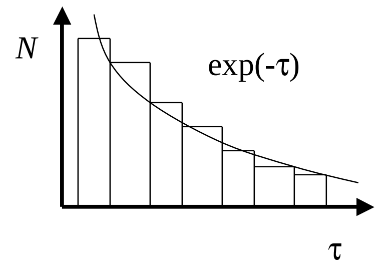
Probability of traveling L before interacting is

$$\begin{aligned} \mathrm{P}(L) &= (1 - n\,\sigma\,l\,/\,N)\,(1 - n\,\sigma\,l\,/\,N)\,(1 - n\,\sigma\,l\,/\,N)\,\ldots \\ &= (1 - n\,\sigma\,l\,/\,N)^N = \exp(-n\,\sigma\,L) \\ \mathrm{P}(L) &= \exp(-\tau) \end{aligned}$$

τ = number of mean free paths over distance L.

Probability Distribution Function

The probability distribution function (PDF) for photons to travel optical depth τ before an interaction is $\exp(-\tau)$. If we pick τ uniformly over the range 0 to infinity we will not reproduce $\exp(-\tau)$. We want to pick lots of small τs and fewer large τs. Same with a scattering phase function: want to get the correct number of photons scattered into different directions, forward and back scattering, etc.



Cumulative Distribution Function

$$\boxed{\mathrm{CDF} = \text{Area under PDF} = \int P(x)\,\mathrm{d}x}$$

Want to randomly choose τ, θ, λ, … so that PDF is reproduced

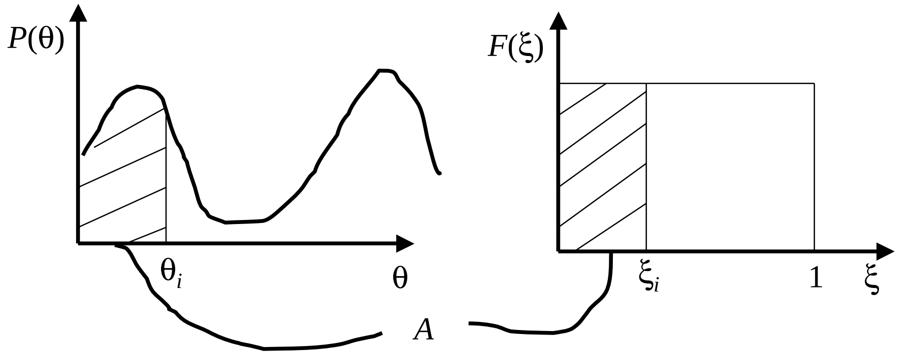
ξ is a random number uniformly chosen in range [0,1]

$$\boxed{\xi = \int_0^X P(x)\mathrm{d}x \Rightarrow X}$$

$$\boxed{\int_{-\infty}^{\infty} P(x)\mathrm{d}x = 1}$$

The above equation is the ***fundamental principle*** behind Monte Carlo techniques and is used to sample randomly from PDFs.

e.g., $P(\theta) = \cos\theta$ and we want to map ξ to θ. Choose random θs to "fill in" $P(\theta)$



$$\boxed{\xi_i = \int_0^{\theta_i} P(\theta)\,\mathrm{d}\theta = \sin\theta_i \Rightarrow \theta_i = \sin^{-1}\xi_i}$$

If we sample many random θ_i in this way and "bin" them, we will reproduce the curve $P(\theta) = \cos\theta$.

Choosing a Random Optical Depth

$P(\tau)$ = exp($-\tau$), i.e., photon travels τ before interaction

$$\boxed{\xi = \int_0^{\tau} \mathrm{e}^{-\tau}\,\mathrm{d}\tau = 1 - \mathrm{e}^{-\tau} \Rightarrow \tau = -\log(1-\xi)}$$

Since ξ is in range [0,1], then $(1-\xi)$ is also in range [0,1], so we may write:

$$\boxed{\tau = -\log\xi}$$

We find the physical distance, L, that the photon has traveled from:

$$\boxed{\tau = \int_0^{L} n\,\sigma\,\mathrm{d}s}$$

Random Isotropic Direction

Solid angle is $d\Omega = \sin\theta \, d\theta \, d\phi$, we want to choose (θ, ϕ) so they fill in PDFs for θ and ϕ. $P(\theta)$ normalized over $[0, \pi]$, $P(\phi)$ normalized over $[0, 2\pi]$:

$$P(\theta) = \tfrac{1}{2} \sin\theta \qquad\qquad P(\phi) = 1 / 2\pi$$

Using fundamental principle from above:

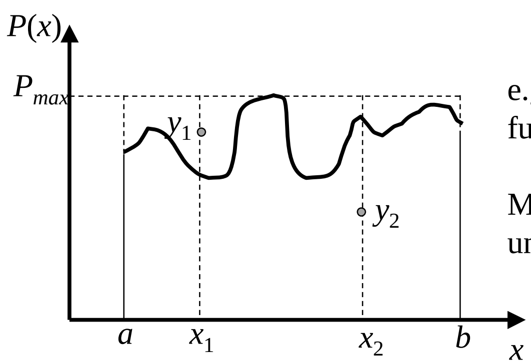
$$\xi = \int_0^\theta P(\theta)\,\mathrm{d}\theta = \frac{1}{2}\int_0^\theta \sin\theta\,\mathrm{d}\theta = \frac{1}{2}(\cos\theta - 1)$$
$$\xi = \int_0^\phi P(\phi)\,\mathrm{d}\phi = \frac{1}{2\pi}\int_0^\phi \mathrm{d}\phi = \frac{\phi - 1}{2\pi}$$

$$\theta = \cos^{-1}(2\xi - 1)$$
$$\phi = 2\pi\xi$$

Use this formula for emitting photons isotropically from a point source, or for choosing a scattering direction for isotropic scattering.

Rejection Method

The rejection method is used when we cannot invert the PDF as in the above examples to obtain analytic formulae for random θ, λ, etc.

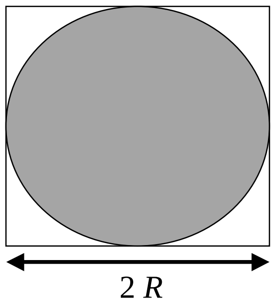


e.g., $P(x)$ can be complicated function or tabulated

Multiply two random numbers: uniform probability / area

Choose x_1 in range $[a, b]$: $x_1 = a + \xi(b - a)$, calculate $P(x_1)$
Choose y_1 in range $[0, P_{max}]$: $y_1 = \xi P_{max}$
If $y_1 > P(x_1)$, reject x_1. Choose new x_2, y_2 until $y_2 < P(x_2)$: accept x_2
Efficiency = Area under $P(x)$

Calculate π by the Rejection Method



Choose N random positions (x_i, y_i):
Choose x_i in range $[-R, R]$: $x_i = (2\xi - 1)\, R$
Choose y_i in range $[-R, R]$: $y_i = (2\xi - 1)\, R$
Reject (x_i, y_i) if $x_i^2 + y_i^2 > R^2$
Number accepted / $N = \pi R^2 / 4R^2$

$$N_A / N = \pi / 4$$

Increase accuracy (signal to noise) by increasing N.

FORTRAN 77:

```
do i = 1, N
   x = 2.*ran – 1.
   y = 2.*ran –1.
   if ( (x*x + y*y) .lt. 1. ) NA = NA + 1
end do
pi = 4.*NA / N
```

Albedo

When photon gets to interaction location at the randomly chosen optical depth, τ, we must decide whether the photon is scattered or absorbed. We use the ***albedo*** or ***scattering probability***. It is the ratio of scattering to total opacity:

$$\boxed{a = \frac{\sigma_S}{\sigma_S + \sigma_A}}$$

To decide if a photon is scattered we choose a random number in the range [0, 1] and scatter if $\xi < a$, otherwise the photon is absorbed.

We now have the tools required to write a Monte Carlo radiation transfer program for isotropic scattering in a constant density slab…