13. Monte Carlo II
Scattering Codes

- Plane parallel scattering atmosphere
- Optical depths & physical distances
- Emergent flux & intensity
- Internal intensity moments

Constant density slab of vertical optical depth $\tau_{\text{max}} = n \sigma z_{\text{max}}$.
Normalized length units $z = z / z_{\text{max}}$.

Emit photons
Photon scatters in slab until: absorbed: terminate, start new photon
  $z < 0$: re-emit photon
  $z > 1$: escapes, “bin” photon

Loop over photons
Sample optical depths, test for absorption, test if photon is in slab
Emitting Photons: Photons need an initial starting location and direction. Isotropic emission from a surface.

Start photon at \((x, y, z) = (0, 0, 0)\)

\[
I_{\nu}(\mu) = \frac{dE}{\mu d\Omega} \Rightarrow \frac{dE}{d\Omega} \propto \frac{dN}{d\Omega} \propto \mu I_{\nu}(\mu)
\]

Sample \(\mu\) from \(P(\mu) = \mu I(\mu)\) using cumulative distribution.

Normalization: emitting outward from lower boundary, so \(0 < \mu < 1\)

\[
\xi = \frac{\int_{0}^{\mu} P(\mu) d\mu}{\int_{0}^{\infty} P(\mu) d\mu} = \mu^2 \Rightarrow \mu = \frac{\sqrt{\xi}}{\sqrt{5}}
\]
Distance Traveled: Random optical depth $\tau = -\log \xi$, and $\tau = n \sigma L$, so distance traveled is:

$$L = \frac{\tau}{\tau_{\max}} z_{\max}$$

Scattering: Assume isotropic scattering, so new photon direction is:

$$\theta = \cos^{-1}(2\xi - 1)$$
$$\phi = 2\pi \xi$$

Absorb or Scatter: Scatter if $\xi < a$, otherwise the photon is absorbed, exit “do while in slab” loop and start a new photon.

Structure of FORTRAN 77 program:

```fortran
do i = 1, nphotons
    call emit_photon
    do while ( (z .ge. 0.) .and. (z .le. 1.)) ! photon is in slab
        L = -log(ran) * zmax / taumax
        z = z + L * nz ! update photon position, x,y,z
        if (ran .lt. albedo) then
            call scatter
        else
            goto 2 ! terminate photon
        end if
    end do
    if (z .le. 0.) goto 1 ! re-start photon
    bin photon according to direction
end do
2 continue ! exit for absorbed photons, start a new photon
end do
```
**Intensity Moments**

The moments of the radiation field are:

\[
J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega \\
H_\nu = \frac{1}{4\pi} \int I_\nu \mu \, d\Omega \\
K_\nu = \frac{1}{4\pi} \int I_\nu \mu^2 \, d\Omega
\]

We may compute these moments throughout the slab. First slit the slab into layers, then tally number of photons, weighted by powers of their direction cosines to obtain \(J, H, K\). Contribution to specific intensity from a single photon is

\[
\Delta I_\nu = \frac{\Delta E}{|\mu| \Delta A \Delta t \Delta \nu \Delta \Omega} = \frac{F_\nu}{|\mu| N_0 \Delta \Omega} = \frac{\pi B_\nu}{|\mu| N_0 \Delta \Omega}
\]

Substitute these in the intensity moment equations and convert the integral to a summation to get:

\[
J_\nu = \frac{B_\nu}{4 N_0} \sum_i \frac{1}{|\mu_i|} \\
H_\nu = \frac{B_\nu}{4 N_0} \sum_i |\mu_i| \\
K_\nu = \frac{B_\nu}{4 N_0} \sum_i |\mu_i|^2
\]

Note the mean flux, \(H\), is just the net energy passing each level: number of photons traveling up minus number traveling down.
Examples

Choose random frequency from a power law spectrum

\[ F(\nu) = \nu^{-\alpha} \text{ with } \nu_1 < \nu < \nu_2 \]

\[
[\frac{\int_{\nu_1}^{\nu_2} \nu^{-\alpha} d\nu}{\nu_1^{1-\alpha} \nu_2^{1-\alpha}}]^{1/1-\alpha} \Rightarrow \nu = (\nu_1^{1-\alpha} + \xi [\nu_2^{1-\alpha} - \nu_1^{1-\alpha}])^{1/1-\alpha}
\]

Examples

Choose random location for emission of photon in a circumstellar shell with emissivity \( j(r) \sim (r/R_\ast)^{-\alpha} \) and \( r \) in the range \( R_\ast < r < R_{\text{max}} \)

\[
[\frac{\int_{R_\ast}^{R_{\text{max}}} \left( \frac{r}{R_\ast} \right)^{-\alpha} dr}{\frac{1}{1-R_\ast / R_{\text{max}}} \left[ 1-R_\ast / R_{\text{max}} \right] \int_{R_\ast}^{R_{\text{max}}} \left( \frac{r}{R_\ast} \right)^{-\alpha} dr}] \Rightarrow r = R_\ast / (1 - \xi [1-R_\ast / R_{\text{max}}])
\]
Examples

Emission from a galaxy disk with stellar emission approximated as a smooth emissivity: \( j(r, z) \sim \exp(-|z|/Z) \exp(-r/R) \). Defined over \( 0 < r < R_{\text{max}}, 0 < z < Z_{\text{max}} \).

Split into separate probabilities for \( r \) and \( z \)

\[
\xi = \frac{\int_{0}^{r} \exp(-r/R) \, dr}{\int_{0}^{R_{\text{max}}} \exp(-r/R) \, dr} = \frac{[1-\exp(-r/R)]}{[1-\exp(-R_{\text{max}}/R)]}
\]

\[ r = -R \log(1 - \xi [1 - \exp(-R_{\text{max}}/R)]) \]

Rejection criteria so no photons emitted inside some inner spherical volume?

Tutorial Example

Choose a random scattering angle from the Rayleigh phase function: \( P(\theta) \sim 1 + \cos^2 \theta \).

First normalize the phase function over \( 0 < \theta < \pi \)

Then use CDF to determine random \( \theta \)