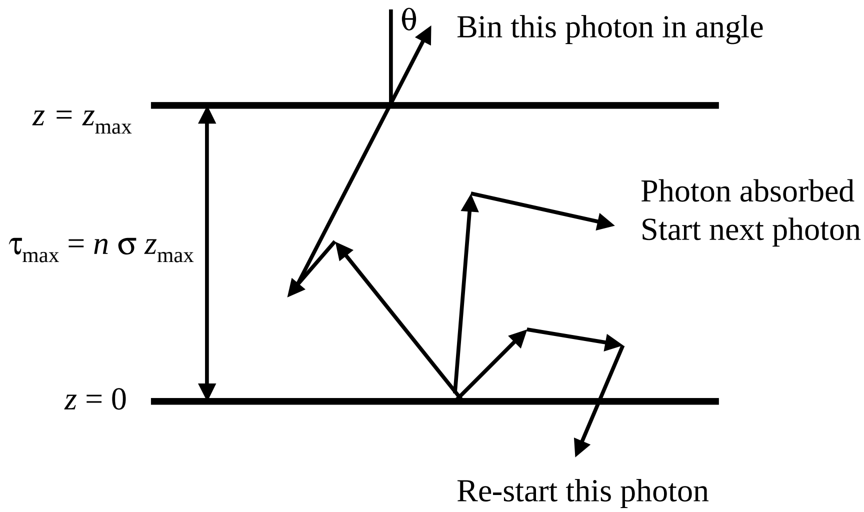
13. Monte Carlo II
Scattering Codes

- Plane parallel scattering atmosphere
- Optical depths & physical distances
- Emergent flux & intensity
- Internal intensity moments

Constant density slab of vertical optical depth $\tau_{max} = n\, \sigma\, z_{max}$.
Normalized length units $z = z / z_{max}$.

Emit photons
Photon scatters in slab until: absorbed: terminate, start new photon
$z < 0$: re-emit photon
$z > 1$: escapes, "bin" photon

Loop over photons
Sample optical depths, test for absorption, test if photon is in slab



Emitting Photons: Photons need an initial starting location and direction. Isotropic emission from a surface.

Start photon at $(x, y, z) = (0, 0, 0)$

$$\boxed{I_\nu(\mu) = \frac{dE}{\mu\, dA\, dt\, d\nu\, d\Omega} \quad \Rightarrow \frac{dE}{dA\, dt\, d\nu\, d\Omega} \propto \frac{dN}{d\Omega} \propto \mu I_\nu(\mu)}$$

Sample μ from $P(\mu) = \mu\, I\,(\mu)$ using cumulative distribution.
Normalization: emitting outward from lower boundary, so $0 < \mu < 1$

$$\boxed{\xi = \frac{\int_0^\mu P(\mu)\,\mathrm{d}\mu}{\int_0^1 P(\mu)\,\mathrm{d}\mu} = \mu^2 \quad \Rightarrow \quad \mu = \sqrt{\xi}}$$

Distance Traveled: Random optical depth $\tau = -\log \xi$, and $\tau = n\,\sigma\,L$, so distance traveled is:

$$\boxed{L = \frac{\tau}{\tau_{\max}} z_{\max}}$$

Scattering: Assume isotropic scattering, so new photon direction is:

$$\boxed{\begin{array}{l}\theta = \cos^{-1}(2\xi - 1) \\ \phi = 2\pi\xi\end{array}}$$

Absorb or Scatter: Scatter if $\xi < a$, otherwise the photon is absorbed, exit “do while in slab” loop and start a new photon.

Structure of FORTRAN 77 program:

```
do i = 1, nphotons
1         call emit_photon
          do while ( (z .ge. 0.) .and. (z .le. 1.) ) ! photon is in slab
                    L = -log(ran) * zmax / taumax
                    z = z + L * nz            ! update photon position, x,y,z
                    if  (ran .lt. albedo) then
                              call scatter
                    else
                              goto 2          ! terminate photon
                    end if
          end do
          if (z .le. 0.) goto 1     ! re-start photon
          bin photon according to direction
2 continue          ! exit for absorbed photons, start a new photon
end do
```

Intensity Moments

The moments of the radiation field are:

$$J_\nu = \frac{1}{4\pi}\int I_\nu \, \mathrm{d}\Omega \quad H_\nu = \frac{1}{4\pi}\int I_\nu \, \mu \, \mathrm{d}\Omega \quad K_\nu = \frac{1}{4\pi}\int I_\nu \, \mu^2 \, \mathrm{d}\Omega$$

We may compute these moments throughout the slab. First slit the slab into layers, then tally number of photons, weighted by powers of their direction cosines to obtain *J*, *H*, *K*. Contribution to specific intensity from a single photon is

$$\Delta I_\nu = \frac{\Delta E}{|\mu| \Delta A \, \Delta t \, \Delta \nu \, \Delta\Omega} = \frac{F_\nu}{|\mu| N_0 \, \Delta\Omega} = \frac{\pi B_\nu}{|\mu| N_0 \, \Delta\Omega}$$

Substitute these in the intensity moment equations and convert the integral to a summation to get:

$$J_\nu = \frac{B_\nu}{4 N_0}\sum_i \frac{1}{|\mu_i|} \quad H_\nu = \frac{B_\nu}{4 N_0}\sum_i \frac{\mu_i}{|\mu_i|} \quad K_\nu = \frac{B_\nu}{4 N_0}\sum_i \frac{\mu_i^2}{|\mu_i|}$$

Note the mean flux, *H*, is just the net energy passing each level: number of photons traveling up minus number traveling down.

Examples

Choose random frequency from a power law spectrum
$F(\nu) \sim \nu^{-\alpha}$ with $\nu_1 < \nu < \nu_2$

$$\xi = \frac{\int_{\nu_1}^{\nu} \nu^{-\alpha}\, \mathrm{d}\nu}{\int_{\nu_1}^{\nu_2} \nu^{-\alpha}\, \mathrm{d}\nu} = \frac{[\nu^{1-\alpha}]_{\nu_1}^{\nu}}{[\nu^{1-\alpha}]_{\nu_1}^{\nu_2}} \quad \Rightarrow \quad \nu = \left(\nu_1^{1-\alpha} + \xi[\nu_2^{1-\alpha} - \nu_1^{1-\alpha}]\right)^{1/(1-\alpha)}$$

Examples

Choose random location for emission of photon in a circumstellar shell with emissivity $j(r) \sim (r / R_*)^{-\alpha}$ and r in the range $R_* < r < R_{max}$

$$\xi = \frac{\int_{R_*}^{r} \left(\frac{r}{R_*}\right)^{-\alpha} \mathrm{d}r}{\int_{R_*}^{R_{\max}} \left(\frac{r}{R_*}\right)^{-\alpha} \mathrm{d}r} = \frac{[1 - R_* / r]}{[1 - R_* / R_{\max}]} \quad \Rightarrow \quad r = R_* / \left(1 - \xi[1 - R_* / R_{\max}]\right)$$

Examples

Emission from a galaxy disk with stellar emission approximated as a smooth emissivity: $j(r, z) \sim \exp(- |z| / Z) \exp(- r / R)$. Defined over $0 < r < R_{max}$, $0 < z < Z_{max}$

Split into separate probabilities for r and z

$$\xi = \frac{\int_0^r \exp(-r/R)\,\mathrm{d}r}{\int_0^{R_{max}} \exp(-r/R)\,\mathrm{d}r} = \frac{[1-\exp(-r/R)]}{[1-\exp(-R_{max}/R)]}$$

$$\Rightarrow \quad r = -R\log\left(1-\xi[1-\exp(-R_{max}/R)]\right)$$

Rejection criteria so no photons emitted inside some inner spherical volume?

Tutorial Example

Choose a random scattering angle from the Rayleigh phase function: $P(\theta) \sim 1 + \cos^2 \theta$.

First normalize the phase function over $0 < \theta < \pi$

Then use CDF to determine random θ