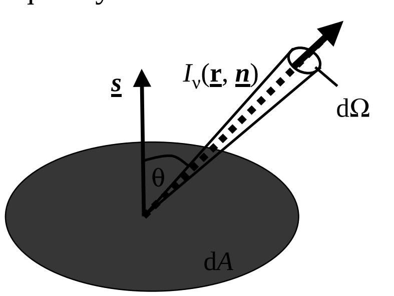
2. Radiation Field Basics I

- Rutten: 2.1
- Basic definitions of intensity, flux
- Energy density, radiation pressure

Specific Intensity

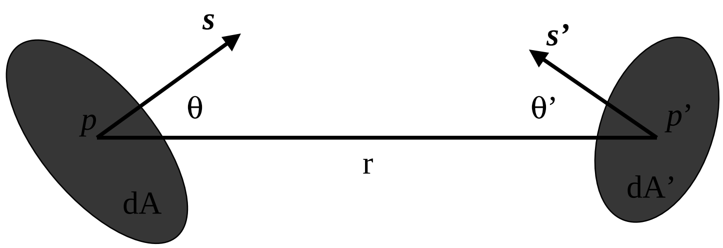
$$\boxed{\mathrm{d}E_\nu = I_\nu \cos\theta \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\nu \, \mathrm{d}\Omega}$$

Pencil beam of radiation at position **r**, direction $\underline{\boldsymbol{n}}$, carrying energy d$E_\nu$, passing through area d$A$, between the times t and t + dt, in the frequency band between ν and ν + dν.



$\underline{\boldsymbol{s}}$ is normal to dA

Units of I_ν: J/m^2/s/Hz/sr (ergs/cm^2/s/Hz/sr)



No sources or sinks of radiation. Pencil beam of radiation passing through dA at p and dA' at p'. Radiant energy passing through both areas is the same:

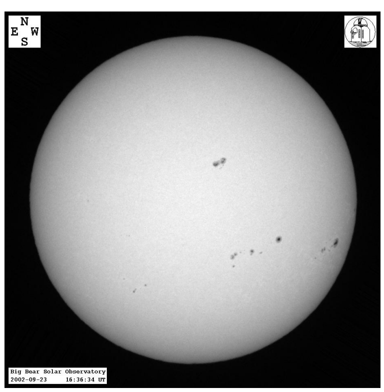
$$\boxed{\mathrm{d}E_\nu = I_\nu \cos\theta\, \mathrm{d}A\, \mathrm{d}t\, \mathrm{d}\nu\, \mathrm{d}\Omega = \mathrm{d}E_\nu^{'} = I_\nu^{'} \cos\theta'\, \mathrm{d}A'\, \mathrm{d}t\, \mathrm{d}\nu\, \mathrm{d}\Omega'}$$

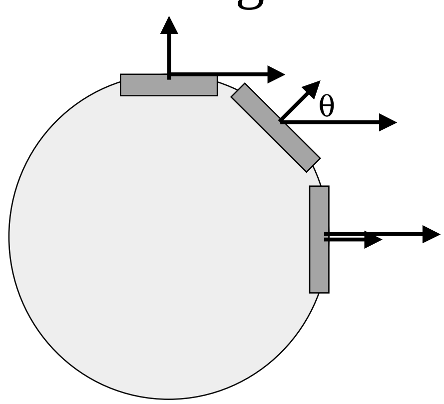
Solid angle dΩ subtended by d*A*' at *p*, dΩ' subtended by d*A* at *p*':

$\boxed{\mathrm{d}\Omega = \mathrm{d}A' \cos\theta' / r^2\ ;\ \mathrm{d}\Omega' = \mathrm{d}A \cos\theta / r^2}$ Therefore: $\boxed{I_\nu = I_\nu^{'}}$

Specific intensity independent of distance when no sources or sinks.

Solar Limb Darkening





Assume plane parallel atmosphere
Measure *I* at different positions on solar disk => get *I*(θ)

Mean Intensity

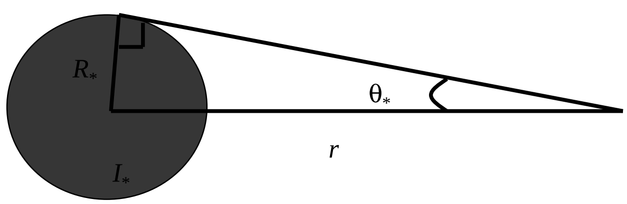
$$J_\nu = \frac{1}{4\pi}\int I_\nu \, \mathrm{d}\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} I_\nu \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

J_ν is a very important quantity. It determines level populations and ionization state throughout atmosphere. Like I_ν, it is a function of position. For a plane parallel atmosphere (no ϕ dependence) with $\mu = \cos\theta$:

$$J_\nu(\mathrm{z}) = \frac{1}{4\pi}\int I_\nu(\mathrm{z},\theta)\, 2\pi \sin\theta \, \mathrm{d}\theta = \frac{1}{2}\int_{-1}^{1} I_\nu(\mathrm{z},\mu)\, \mathrm{d}\mu$$

We'll later use *moments of the radiation field*. J_ν is the 0^{th} moment

Moment operator M operating on f: $\mathrm{M}^{(n)}[f] = \frac{1}{2}\int_{-1}^{1} f \, \mu^n \, \mathrm{d}\mu$



What is J_ν at distance r from a star with uniform specific intensity I_* across its surface?

$I = I_*$	for	$0 < \theta < \theta_*$	$(\mu_* < \mu < 1)$
$I = 0$	for	$\theta > \theta_*$	$(\mu < \mu_*)$

$$J = \frac{1}{2}\int_{\mu_*}^{1} I \, \mathrm{d}\mu = \tfrac{1}{2} I_*(1-\mu_*)$$

$$J = I_* \tfrac{1}{2}\left(1-\sqrt{1-R_*^2/r^2}\right) = w I_*$$

w is the *dilution factor*

At large r, $w = R^2/4r^2$

Flux

The flux enables us to calculate the ***total energy, E***, passing through a surface in a given time, i.e., integrated over all directions. The energy transport can be positive or negative.

$$E = \int \mathrm{d}E_\nu \, \mathrm{d}\Omega = \mathrm{d}\nu \, \mathrm{d}t \int I_\nu(\underline{r}, \underline{n}, t) \, \underline{\hat{n}} \bullet \underline{\mathrm{d}A} \, \mathrm{d}\Omega$$

Monochromatic Flux, $\mathcal{F}_\nu$: The net flow of radiant energy per second through an area dA in time dt in frequency range dν.

$$\mathcal{F}_\nu = \int I_\nu \cos\theta \, \mathrm{d}\Omega = \int_0^{2\pi} \int_0^{\pi} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

This is used for specifying the energetics of radiation through stellar interiors, atmospheres, ISM, etc. In principle, flux is a vector.

In stellar atmospheres, the outward radial direction is always implied positive, so that

$$\begin{aligned}
\mathcal{F}_\nu(z) &= \int_0^{2\pi} \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi + \int_0^{2\pi} \int_{\pi/2}^{\pi} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \int_0^{2\pi} \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi - \int_0^{2\pi} \int_{\pi/2}^{\pi} I_\nu(\pi - \theta) \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&\equiv \mathcal{F}_\nu^+(z) - \mathcal{F}_\nu^-(z)
\end{aligned}$$

With both the outward flux, $\mathcal{F}_\nu^+$, and the inward flux, $\mathcal{F}_\nu^-$, positive. Isotropic radiation has $\mathcal{F}_\nu^+ = \mathcal{F}_\nu^- = \pi I_\nu$ and $\mathcal{F}_\nu = 0$. Axisymmetry:

$$\begin{aligned}
\mathcal{F}_\nu(z) &= 2\pi \int_0^{\pi} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= 2\pi \int_0^{1} I_\nu \, \mu \, \mathrm{d}\mu - 2\pi \int_0^{-1} I_\nu \, \mu \, \mathrm{d}\mu \\
&= \mathcal{F}_\nu^+(z) - \mathcal{F}_\nu^-(z)
\end{aligned}$$

The flux emitted by a star per unit area of its surface is $\mathcal{F}_\nu = \mathcal{F}_\nu^+ = \pi I_\nu^*$ where I_ν^* is the intensity, averaged over the apparent stellar disk, received by an observer. This equality is why that flux is often written as $\pi F = \mathcal{F}$, so that $F = I^*$, with F called the ***Astrophysical Flux***.

This explains the often confusing factors of π that are floating about in definitions of flux:
$\mathcal{F}$ = ***Monochromatic Flux*** or just the Flux; F = ***Astrophysical Flux***
They are related by $\pi F = \mathcal{F}$.

In terms of moments of the radiation field, the first moment is defined as the ***Eddington Flux***, H_ν. For plane parallel geometry:

$$\boxed{H_\nu \equiv \frac{1}{4\pi}\int I_\nu \cos\theta \, d\Omega = \frac{\mathcal{F}_\nu}{4\pi} = \frac{F_\nu}{4} = \frac{1}{2}\int_{-1}^{1} I_\nu \, \mu \, d\mu}$$

Stellar Luminosity

Flux = energy/second per area
Luminosity = energy/second

$$\boxed{L_\nu = \mathcal{F}_\nu A_* = 4\pi R_*^2 \, \pi I_\nu}$$

Assume $I_\nu = B_\nu$ and integrate:

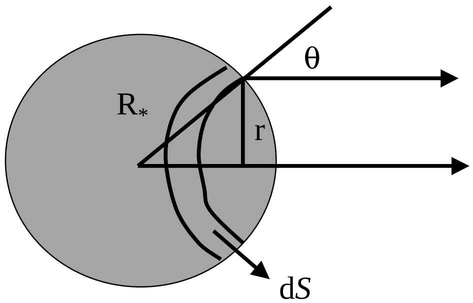
$$\boxed{L = \int L_\nu \, d\nu = 4\pi R_*^2 \, \pi \int B_\nu \, d\nu = 4\pi R_*^2 \, \sigma T^4}$$

Unresolved Sources

Relate energy observed to $\mathcal{F}_\nu$ at stellar surface:

Energy received per detector area, from anulus: $\boxed{\mathrm{d}f_\nu = I_\nu \, \mathrm{d}\omega}$

$\mathrm{d}\omega$ = solid angle of anulus



Anulus area ($r = R_* \sin\theta$):

$$\boxed{\mathrm{d}S = 2\pi\, r\, \mathrm{d}r = 2\pi\, R_*^2\, \mu\, \mathrm{d}\mu}$$

$\mathrm{d}\omega = \mathrm{d}S / D^2$

Integrate over ω:

$$\boxed{\begin{aligned} f_\nu &= 2\pi (R_*/D)^2 \int_0^1 I(R_*, \mu, \nu)\mu\, \mathrm{d}\mu \\ &= (R_*/D)^2\, \mathcal{F}(R_*, \nu) \\ &= \frac{1}{4} \alpha_*^2\, \mathcal{F}(R_*, \nu) \end{aligned}}$$

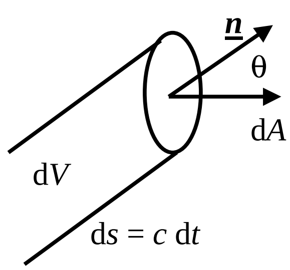
α_* = angular diameter

Unresolved => measure flux

Inverse square law. Know α_* , get absolute flux at star

Energy Density

The energy flow in a beam of radiation is $\boxed{\mathrm{d}E_\nu = I_\nu \cos\theta\, \mathrm{d}A\, \mathrm{d}t\, \mathrm{d}\nu\, \mathrm{d}\Omega}$



The flow has velocity c (photons) and travels a distance ds in time dt = ds/c through volume dV = dA ds cos θ. Thus, each beam carries $\mathrm{d}E_\nu = (1/c)\, I_\nu\, \mathrm{d}\Omega\, \mathrm{d}V$. If multiple beams pass through a small volume ΔV, integration over ΔV and over all beam directions gives the radiant energy $E_\nu\, \mathrm{d}\nu$ contained in ΔV across bandwidth dν as:

$$\boxed{E_\nu \mathrm{d}\nu = \frac{1}{c} \int_{\Delta V} \int_{\Omega} I_\nu \;\mathrm{d}V\, \mathrm{d}\Omega\, \mathrm{d}\nu}$$

For sufficiently small ΔV, the intensity is homogeneous, so the two integrations (V, Ω) are independent. The energy density is

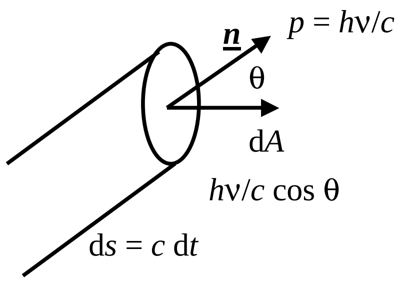
$$\boxed{u_\nu = \frac{1}{c}\int I_\nu \, \mathrm{d}\Omega}$$

Radiation Pressure

Each photon has momentum $p = h\nu/c$. Component of momentum normal to a solid wall per time per area is

$$\boxed{\mathrm{d}p_\nu = \frac{1}{c}\frac{\mathrm{d}E_\nu \cos\theta}{\mathrm{d}A\,\mathrm{d}t}}$$

Re-write in terms of I_ν and integrating over solid angle gives:



$$\boxed{p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta \, \mathrm{d}\Omega}$$

Isotropic radiation has $p_\nu = u_\nu/3$. Radiation pressure is analogous to gas pressure, being the pressure of the photon gas.