2. Radiation Field Basics I

- Rutten: 2.1
- Basic definitions of intensity, flux
- Energy density, radiation pressure

Specific Intensity

\[ dE_\nu = I_\nu \cos \theta dA dt d\nu d\Omega \]

Pencil beam of radiation at position \( \mathbf{r} \), direction \( \mathbf{n} \), carrying energy \( dE_\nu \), passing through area \( dA \), between the times \( t \) and \( t + dt \), in the frequency band between \( \nu \) and \( \nu + d\nu \).

- \( \mathbf{s} \) is normal to \( dA \)
- Units of \( I_\nu \): J/m\(^2\)/s/Hz/sr
  (ergs/cm\(^2\)/s/Hz/sr)
No sources or sinks of radiation. Pencil beam of radiation passing through dA at p and dA’ at p’. Radiant energy passing through both areas is the same:

\[
dE_v = I_v \cos \theta \, dA \, d\nu \, d\Omega = dE_v' = I_v' \cos \theta' \, dA' \, d\nu \, d\Omega'
\]

Solid angle d\Omega subtended by dA’ at p, d\Omega’ subtended by dA at p’:

\[
d\Omega = dA' \cos \theta' / r^2 ; \quad d\Omega' = dA \cos \theta / r^2
\]

Therefore: \[ I_v = I_v' \]

Specific intensity independent of distance when no sources or sinks.

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**Solar Limb Darkening**

Assume plane parallel atmosphere
Measure I at different positions on solar disk => get I(\theta)
Mean Intensity

\[ J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi I_\nu \sin \theta \, d\theta \, d\phi \]

\( J_\nu \) is a very important quantity. It determines level populations and ionization state throughout atmosphere. Like \( I_\nu \), it is a function of position. For a plane parallel atmosphere (no \( \phi \) dependence) with \( \mu = \cos \theta \):

\[ J_\nu(z) = \frac{1}{4\pi} \int I_\nu(z, \theta) 2\pi \sin \theta \, d\theta \, d\phi = \frac{1}{2} \int_{-1}^1 I_\nu(z, \mu) \, d\mu \]

We’ll later use moments of the radiation field. \( J_\nu \) is the 0th moment

Moment operator \( M \) operating on \( f \):

\[
M^{(n)}[f] = \frac{1}{2} \int_{-1}^1 f \, \mu^n \, d\mu
\]

What is \( J_\nu \) at distance \( r \) from a star with uniform specific intensity \( I_* \) across its surface?

\[
I = I_* \quad \text{for} \quad 0 < \theta < \theta_* \quad (\mu_* < \mu < 1)
\]

\[
I = 0 \quad \text{for} \quad \theta > \theta_* \quad (\mu < \mu_*)
\]

\[
J = \frac{1}{2} \int_{\mu_*}^1 I \, d\mu = \frac{1}{2} I_* (1 - \mu_*)
\]

\[ J = I_* \frac{1}{2} \left( 1 - \sqrt{1 - R_*/r^2} \right) = w I_* \]

\( w \) is the dilution factor

At large \( r \), \( w = R^2/4r^2 \)
Flux

The flux enables us to calculate the total energy, $E$, passing through a surface in a given time, i.e., integrated over all directions. The energy transport can be positive or negative.

$$E = \int dE_v \, d\Omega = d\nu \, dt \int I_v(r, \theta, \phi) \hat{n} \cdot dA \, d\Omega$$

**Monochromatic Flux, $F_v$:** The net flow of radiant energy per second through an area $dA$ in time $dt$ in frequency range $d\nu$.

$$F_v = \int I_v \cos \theta \, d\Omega = \int_0^{2\pi} \int_0^{\pi} I_v \cos \theta \sin \theta \, d\theta \, d\phi$$

This is used for specifying the energetics of radiation through stellar interiors, atmospheres, ISM, etc. In principle, flux is a vector.

In stellar atmospheres, the outward radial direction is always implied positive, so that

$$F_v(z) = \int_0^{\pi/2} I_v \cos \theta \sin \theta \, d\theta \, d\phi + \int_0^{\pi} I_v \cos \theta \sin \theta \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} I_v \cos \theta \sin \theta \, d\theta \, d\phi - \int_0^{\pi/2} I_v \cos \pi \theta \cos \theta \, d\theta \, d\phi$$

$$\equiv F_v^+(z) - F_v^-(z)$$

With both the outward flux, $F_v^+$, and the inward flux, $F_v^-$, positive. Isotropic radiation has $F_v^+ = F_v^- = \pi \nu$ and $F_v = 0$. Axisymmetry:

$$F_v(z) = 2\pi \int_0^{\pi} I_v \cos \theta \sin \theta \, d\theta \, d\phi$$

$$= 2\pi \int_0^1 \nu \mu \, d\mu - 2\pi \int_0^1 \nu \mu \, d\mu$$

$$= F_v^+(z) - F_v^-(z)$$
The flux emitted by a star per unit area of its surface is \( \mathcal{F}_\nu = \mathcal{F}_{\nu} + \pi I^*_\nu \) where \( I^*_\nu \) is the intensity, averaged over the apparent stellar disk, received by an observer. This equality is why that flux is often written as \( \pi F = \mathcal{F} \), so that \( F = I' \), with \( F \) called the Astrophysical Flux.

This explains the often confusing factors of \( \pi \) that are floating about in definitions of flux:

\( \mathcal{F} = \textit{Monochromatic Flux} \) or just the Flux; \( F = \textit{Astrophysical Flux} \)

They are related by \( \pi F = \mathcal{F} \).

In terms of moments of the radiation field, the first moment is defined as the Eddington Flux, \( H_\nu \). For plane parallel geometry:

\[
H_\nu = \frac{1}{4\pi} \int I_\nu \cos \theta \, d\Omega = \frac{\mathcal{F}_\nu}{4\pi} = \frac{F_\nu}{4} = \frac{1}{2} \int_{-1}^{1} I_\nu \mu \, d\mu
\]

**Stellar Luminosity**

Flux = energy/second per area  
Luminosity = energy/second

\[
L_\nu = \mathcal{F}_\nu A_s = 4\pi R_s^2 \pi I_\nu
\]

Assume \( I_\nu = B_\nu \) and integrate:

\[
L = \int L_\nu \, d\nu = 4\pi R_s^2 \pi \int B_\nu \, d\nu = 4\pi R_s^2 \sigma T^4
\]
Unresolved Sources

Relate energy observed to \( f_\nu \) at stellar surface:

Energy received per detector area, from anulus:

\[
df_\nu = I_\nu \ d\omega
\]

\( d\omega = \text{solid angle of anulus} \)

Anulus area \((r = R_\ast \sin \theta)\):

\[
dS = 2\pi \ r \ dr = 2\pi \ R_\ast^2 \ \mu \ d\mu
\]

\( d\omega = dS / D^2 \)

Integrate over \( \omega \):

\[
\begin{align*}
f_\nu &= 2\pi (R_\ast / D)^3 \int_0^1 I(R_\ast, \mu, \nu) \mu \ d\mu \\
&= (R_\ast / D)^3 \ \mathcal{F}(R_\ast, \nu) \\
&= \frac{1}{4} \alpha_\ast^3 \ \mathcal{F}(R_\ast, \nu)
\end{align*}
\]

\( \alpha_\ast = \text{angular diameter} \)

Unresolved \(\Rightarrow\) measure flux

Inverse square law. Know \( \alpha_\ast \), get absolute flux at star

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Energy Density

The energy flow in a beam of radiation is

\[
dE_\nu = I_\nu \cos \theta \ dA \ dt \ d\nu \ d\Omega
\]

The flow has velocity \( c \) (photons) and travels a distance \( ds = ds/c \) through volume \( dV = dA \ ds \cos \theta \). Thus, each beam carries \( dE_\nu = (1/c) I_\nu \ d\Omega \ dV \). If multiple beams pass through a small volume \( \Delta V \), integration over \( \Delta V \) and over all beam directions gives the radiant energy \( E_\nu \ d\nu \) contained in \( \Delta V \) across bandwidth \( d\nu \) as:

\[
E_\nu \ d\nu = \frac{1}{c} \int_{\Delta V} \int_\Omega I_\nu \ dV \ d\Omega \ d\nu
\]
For sufficiently small $\Delta V$, the intensity is homogeneous, so the two integrations ($V, \Omega$) are independent. The energy density is

$$u_v = \frac{1}{c} \int I_v \, d\Omega$$

**Radiation Pressure**

Each photon has momentum $p = h\nu/c$. Component of momentum normal to a solid wall per time per area is

$$dp_v = \frac{1}{c} \frac{dE_v \cos \theta}{c} \, dA \, dt$$

Re-write in terms of $I_v$ and integrating over solid angle gives:

$$p_v = \frac{1}{c} \int I_v \cos^2 \theta \, d\Omega$$

Isotropic radiation has $p_v = u_v/3$. Radiation pressure is analogous to gas pressure, being the pressure of the photon gas.