3. Radiation Field Basics II

- Rutten: 2.1, 2.2
- Intensity moments
- Emission, extinction, source function
- Equation of radiation transfer
- Optical depth

Moments of the Radiation Field

We have defined the moment operator **M** operating on *f*:

$$\boxed{\mathrm{M}^{(n)}[f] = \frac{1}{2}\int_{-1}^{1} f\,\mu^n\,\mathrm{d}\mu}$$

The first three moments of the specific intensity are historically named *J* (zeroth moment), *H* (first), and *K* (second):

$$J_\nu = \frac{1}{4\pi}\int I_\nu\,\mathrm{d}\Omega$$

$$H_\nu = \frac{1}{4\pi}\int I_\nu \cos\theta\,\mathrm{d}\Omega$$

$$K_\nu = \frac{1}{4\pi}\int I_\nu \cos^2\theta\,\mathrm{d}\Omega$$

For a plane parallel atmosphere (no ϕ dependence), we get

$$\boxed{\begin{aligned} J_\nu(z) &= \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)\,\mathrm{d}\mu \\ H_\nu(z) &= \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)\,\mu\,\mathrm{d}\mu \\ K_\nu(z) &= \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)\,\mu^2\,\mathrm{d}\mu \end{aligned}}$$

These moments are used in solving the equation of radiation transfer. As shown previously, physically *J* and *H* are related to the mean intensity of radiation and the flux respectively. *K* is related to the radiation pressure by

$$\boxed{p_\nu = \frac{4\pi}{c} K_\nu}$$

Photon Interactions

- Scattering: change direction (energy slightly)
- Absorption: energy added to K.E. of particles: photon thermalized
- Emission: energy taken from thermal energy of particles

Emission Coefficient

j_ν defined by: $$\boxed{\mathrm{d}E_\nu \equiv j_\nu \,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega}$$

Energy dE_ν added to radiation in the volume dV in the frequency range dν in time dt in direction about solid angle dΩ. Energy added can be due to stimulated emission, spontaneous emission, thermal emission, or energy scattered into the beam. The intensity contribution to a beam from local emission along a path length ds is:

$$\boxed{\mathrm{d}I_\nu(s) = j_\nu(s)\,\mathrm{d}s}$$

Extinction Coefficient

The energy removed from a beam along a path length ds defines the extinction coefficient. It may be defined per particle, per mass, or per volume

$$\boxed{\mathrm{d}I_\nu(s) = -I_\nu\,\sigma_\nu\,n\,\mathrm{d}s}$$

σ_ν is the monochromatic extinction coefficient or cross section per particle (m^2) and n the particle density (m^{-3}). The definition per path length is

$$\boxed{\mathrm{d}I_\nu(s) = -I_\nu\,\alpha_\nu\,\mathrm{d}s}$$

where α_ν is in units of m^{-1}. The definition per mass is

$$\boxed{\mathrm{d}I_\nu(s) = -I_\nu\,\kappa_\nu\,\rho\,\mathrm{d}s}$$

where κ_ν has units m^2 kg^{-1} and ρ is the density (kg m^{-3}).

Source Function

A very important quantity in radiation transfer theory, whose name will become clear soon, is the ***Source Function***. It is has the same units as intensity and is defined:

$$\boxed{S_\nu \equiv \frac{j_\nu}{\alpha_\nu}}$$

When multiple processes contribute to the local emission and extinction, the total source function is:

$$\boxed{S_\nu^{\mathrm{tot}} = \frac{\sum j_\nu}{\sum \alpha_\nu}}$$

For example, in a spectral line:

$$\boxed{S_\nu^{\mathrm{tot}} = \frac{j_\nu^c + j_\nu^l}{\alpha_\nu^c + \alpha_\nu^l} = \frac{S_\nu^c + \eta_\nu S_\nu^l}{1+\eta_\nu}}$$

with $\eta_\nu = \alpha_\nu^l / \alpha_\nu^c$ the line-to-continuum extinction ratio; S_ν^c, S_ν^l the continuum and line source functions.

Equation of Radiation Transfer

The equation of radiation transfer (ERT) *along a ray* is:

$$\boxed{\mathrm{d}I_\nu(s) = I_\nu(s+\mathrm{d}s) - I_\nu(s) = j_\nu(s)\,\mathrm{d}s - \alpha_\nu I_\nu(s)\,\mathrm{d}s}$$

Where s is measured along the path in the propagation direction. It can be re-written as follows:

$$\boxed{\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu} \qquad \boxed{\frac{\mathrm{d}I_\nu}{\alpha_\nu\,\mathrm{d}s} = S_\nu - I_\nu}$$

The RHS is just the radiation created (or scattered into the beam), minus radiation destroyed (absorbed or scattered out of the beam). Simply put, RHS = ***sources – sinks.*** Hence, S_ν is the source function. ERT says photons do not decay spontaneously so that I_ν along a ray doesn't change unless photons are added or taken from the beam.

Optical Depth

The monochromatic optical depth, τ_ν, of a medium along a particular direction is:

$$\boxed{\tau_\nu = \int_0^s \alpha_\nu \, ds = \int_0^s \rho\kappa_\nu \, ds}$$

τ_ν is a function of frequency through the opacity: at different frequencies photons see a different opacity and hence optical depth. For non-uniform densities, τ_ν is a function of direction. Physically τ_ν is the number of photon mean free paths through the medium in the given direction. In differential form:

$$\boxed{d\tau_\nu = \alpha_\nu(s)\,ds = \rho(s)\kappa_\nu \, ds}$$

Equation of Radiation Transfer

The ERT along a ray can now be written: $\boxed{\dfrac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu}$

With solution: $\boxed{I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t_\nu)\,e^{-(\tau_\nu - t_\nu)}\,dt_\nu}$

For a homogeneous medium where S_ν does not vary with location:

$$\boxed{I_\nu(D) = I_\nu(0)\,e^{-\tau_\nu(D)} + S_\nu\left(1 - e^{-\tau_\nu(D)}\right)}$$

For optically thick media: $\boxed{I_\nu(D) \approx S_\nu}$

For optically thin: $\boxed{I_\nu(D) \approx I_\nu(0) + [S_\nu - I_\nu(0)]\tau_\nu(D)}$