3. Radiation Field Basics II

- Rutten: 2.1, 2.2
- Intensity moments
- Emission, extinction, source function
- Equation of radiation transfer
- Optical depth

Moments of the Radiation Field

We have defined the moment operator $\mathbf{M}$ operating on $f$:

$$M^{(n)}[f] = \frac{1}{2} \int_{-1}^{1} f \mu^n d\mu$$

The first three moments of the specific intensity are historically named $J$ (zeroth moment), $H$ (first), and $K$ (second):

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega$$

$$H_\nu = \frac{1}{4\pi} \int I_\nu \cos \theta d\Omega$$

$$K_\nu = \frac{1}{4\pi} \int I_\nu \cos^2 \theta d\Omega$$
For a plane parallel atmosphere (no $\phi$ dependence), we get

\[
J_{\nu}(z) = \frac{1}{2} \int_{-1}^{1} I_{\nu}(z, \mu) \, d\mu
\]
\[
H_{\nu}(z) = \frac{1}{2} \int_{-1}^{1} I_{\nu}(z, \mu) \mu \, d\mu
\]
\[
K_{\nu}(z) = \frac{1}{2} \int_{-1}^{1} I_{\nu}(z, \mu) \mu^2 \, d\mu
\]

These moments are used in solving the equation of radiation transfer. As shown previously, physically $J$ and $H$ are related to the mean intensity of radiation and the flux respectively. $K$ is related to the radiation pressure by

\[
p_{\nu} = \frac{4\pi}{c} K_{\nu}
\]

### Photon Interactions

- Scattering: change direction (energy slightly)
- Absorption: energy added to K.E. of particles: photon thermalized
- Emission: energy taken from thermal energy of particles
Emission Coefficient

\( j_\nu \) defined by:

\[ \text{d}E_\nu \equiv j_\nu \text{d}V \text{d}t \text{d}\nu \text{d}\Omega \]

Energy \( \text{d}E_\nu \) added to radiation in the volume \( \text{d}V \) in the frequency range \( \text{d}\nu \) in time \( \text{d}t \) in direction about solid angle \( \text{d}\Omega \). Energy added can be due to stimulated emission, spontaneous emission, thermal emission, or energy scattered into the beam. The intensity contribution to a beam from local emission along a path length \( \text{d}s \) is:

\[ \text{d}I_\nu (s) = j_\nu (s) \text{d}s \]

Extinction Coefficient

The energy removed from a beam along a path length \( \text{d}s \) defines the extinction coefficient. It may be defined per particle, per mass, or per volume

\[ \text{d}I_\nu (s) = -I_\nu \sigma_\nu n \text{d}s \]

\( \sigma_\nu \) is the monochromatic extinction coefficient or cross section per particle (m\(^2\)) and \( n \) the particle density (m\(^{-3}\)). The definition per path length is

\[ \text{d}I_\nu (s) = -I_\nu \alpha_\nu \text{d}s \]

where \( \alpha_\nu \) is in units of m\(^{-1}\). The definition per mass is

\[ \text{d}I_\nu (s) = -I_\nu \kappa_\nu \rho \text{d}s \]

where \( \kappa_\nu \) has units m\(^2\) kg\(^{-1}\) and \( \rho \) is the density (kg m\(^{-3}\)).
**Source Function**

A very important quantity in radiation transfer theory, whose name will become clear soon, is the **Source Function**. It is has the same units as intensity and is defined:

\[ S_\nu = \frac{j_\nu}{\alpha_\nu} \]

When multiple processes contribute to the local emission and extinction, the total source function is:

\[ S_{\nu}^{\text{tot}} = \frac{\sum j_\nu}{\sum \alpha_\nu} \]

For example, in a spectral line:

\[ S_{\nu}^{\text{tot}} = \frac{j_\nu^c + j_\nu^l}{\alpha_\nu^c + \alpha_\nu^l} = \frac{S_\nu^c + \eta_\nu S_\nu^l}{1 + \eta_\nu} \]

with \( \eta_\nu = \alpha_\nu^c/\alpha_\nu^l \) the line-to-continuum extinction ratio; \( S_\nu^c, S_\nu^l \) the continuum and line source functions.

**Equation of Radiation Transfer**

The equation of radiation transfer (ERT) *along a ray* is:

\[
\frac{dI_\nu}{ds} = I_\nu(s + ds) - I_\nu(s) = j_\nu(s)ds - \alpha_\nu I_\nu(s)ds
\]

Where \( s \) is measured along the path in the propagation direction. It can be re-written as follows:

\[
\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu \quad \text{and} \quad \frac{dI_\nu}{\alpha_\nu ds} = S_\nu - I_\nu
\]

The RHS is just the radiation created (or scattered into the beam), minus radiation destroyed (absorbed or scattered out of the beam). Simply put, RHS = *sources – sinks*. Hence, \( S_\nu \) is the source function. ERT says photons do not decay spontaneously so that \( I_\nu \) along a ray doesn’t change unless photons are added or taken from the beam.
Optical Depth

The monochromatic optical depth, $\tau_\nu$, of a medium along a particular direction is:

$$\tau_\nu = \int_0^s \alpha_\nu \, ds = \int_0^s \rho \kappa_\nu \, ds$$

$\tau_\nu$ is a function of frequency through the opacity: at different frequencies photons see a different opacity and hence optical depth. For non-uniform densities, $\tau_\nu$ is a function of direction. Physically $\tau_\nu$ is the number of photon mean free paths through the medium in the given direction. In differential form:

$$d \tau_\nu = \alpha_\nu(s) \, ds = \rho(s) \kappa_\nu(s) \, ds$$

Equation of Radiation Transfer

The ERT along a ray can now be written:

$$\frac{dl_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

With solution:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t_\nu)e^{-(\tau_\nu-t_\nu)} \, dt_\nu$$

For a homogeneous medium where $S_\nu$ does not vary with location:

$$I_\nu(D) = I_\nu(0)e^{-\tau_\nu(D)} + S_\nu\left(1 - e^{-\tau_\nu(D)}\right)$$

For optically thick media:

$$I_\nu(D) \approx S_\nu$$

For optically thin:

$$I_\nu(D) \approx I_\nu(0) + \left[S_\nu - I_\nu(0)\right]\tau_\nu(D)$$