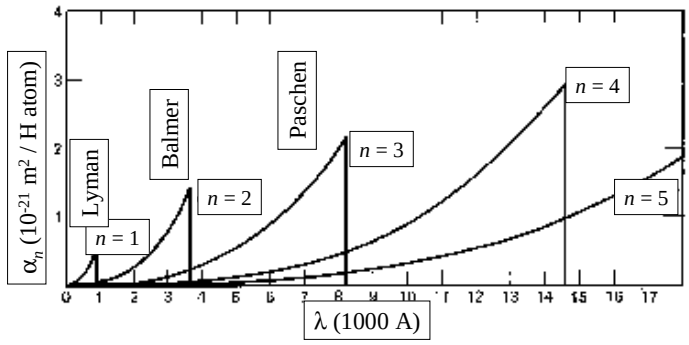
5. Formal Solution of ERT

- Ingredients to produce theoretical spectrum
- Rutten: 4.1
- Formal solution of plane-parallel ERT
- Exponential integrals and operators

Producing a Theoretical Spectrum

- Pure hydrogen atmosphere, $T_{\mathrm{eff}} = 5600$ K
- Assume opacity independent of depth
- Determine wavelength dependent opacity
- Determine temperature structure
- Determine emergent intensity in optical range 3000 A $< \lambda <$ 8600 A

HI Continuous Extinction



For $912 < \lambda < 3646$ can ionize out of $n > 1$: need N_2, N_3, N_4,…
For $3646 < \lambda < 8206$ can ionize out of $n > 2$: need N_3, N_4 ,…
For $\lambda > 8206$ can ionize out of $n > 3$: need N_4 ,…

Hydrogen Bound-Free Opacity

- Level populations for $T_{\mathrm{eff}} = 5600$ K
- Use Boltzmann equation to get populations of H levels $n = 2$, 3, and 4
- Use Kramer's opacity formula to get wavelength dependence

Calculate Level Populations

Use Boltzmann distribution, $g_n = n^2$:

$$\frac{N_n}{N_1} = \frac{g_n}{g_1} \exp[-(\chi_1 - \chi_n)/kT]$$
$$= n^2 \exp[-\Delta E_{1n}/kT]$$
$$= n^2\, 10^{[-\Delta E_{1n}(\mathrm{eV})5040/T]}$$

Use Rydberg formula: $\Delta E_{1n} = \chi\,(1 - 1/n^2)$, $\chi = 13.6$ eV

$$N_2/N_1 = 4\times 10^{[-10.2\times 0.9]} = 2.6\times 10^{-9}$$
$$N_3/N_1 = 9\times 10^{[-11.9\times 0.9]} = 1.2\times 10^{-10}$$
$$N_4/N_1 = 16\times 10^{[-12.75\times 0.9]} = 5.4\times 10^{-11}$$

Calculate Opacity

From L3, Kramer's opacity for hydrogen b-f cross-section (m^2)

$$\sigma_\nu^{\mathrm{bf}} = 2.815\times 10^{25}\frac{Z^4}{n^5\nu^3} g_{\mathrm{bf}} \quad \text{for } \nu \ge \nu_0$$
$$= 1.044\times 10^{-30}\,\lambda^3/n^5 = \sigma_0\,\lambda^3/n^5$$

With λ in A. Total opacity = sum of absorption coefficients from all levels that can be ionized by photon at given λ, times population of level

$$\alpha_\nu^{\mathrm{bf}} = N_2\sigma_2 + N_3\sigma_3 + N_4\sigma_4 \quad \text{for } \lambda \le 3647\mathrm{A}$$
$$= N_3\sigma_3 + N_4\sigma_4 \quad \text{for } 3648\,\mathrm{A} \le \lambda \le 8206\mathrm{A}$$
$$= N_4\sigma_4 \quad \text{for } \lambda \ge 8207\mathrm{A}$$

Ignore contributions from levels $n > 4$

Calculate Opacity

For λ < 3646 A there's enough energy to ionize out of $n > 1$:

$$\begin{aligned}\alpha_\nu^{\mathrm{bf}} &= N_2\,\sigma_2 + N_3\,\sigma_3 + N_4\,\sigma_4 \\ &= N_1\,\sigma_0\,\lambda^3(1/32\,N_2/N_1 + 1/243\,N_3/N_1 + 1/1024\,N_4/N_1) \\ &= 8.2\times10^{-11}\,N_1\,\sigma_0\,\lambda^3\end{aligned}$$

For 3647 A < λ < 8206 A, can ionize out of $n > 2$

$$\begin{aligned}\alpha_\nu^{\mathrm{bf}} &= N_1\,\sigma_0\,\lambda^3(1/243\,N_3/N_1 + 1/1024\,N_4/N_1) \\ &= 5.5\times10^{-13}\,N_1\,\sigma_0\,\lambda^3\end{aligned}$$

For λ > 8207 A, can ionize out of $n > 3$

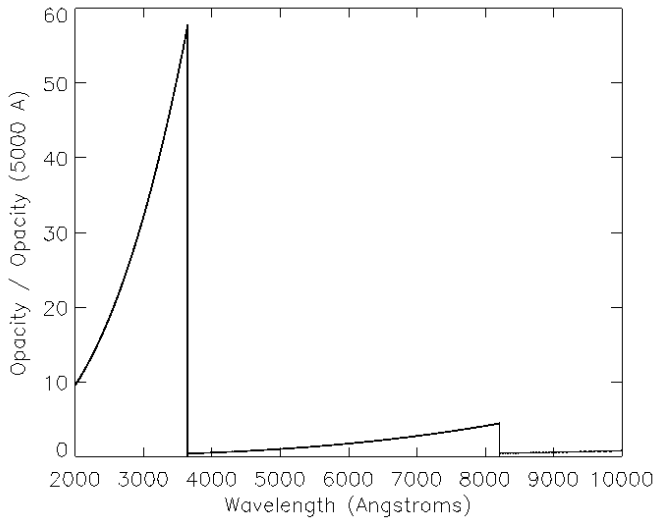
$$\alpha_\nu^{\mathrm{bf}} = N_1\,\sigma_0\,\lambda^3(1/1024\,N_4/N_1) = 5.3\times10^{-14}\,N_1\,\sigma_0\,\lambda^3$$

Calculate Opacity

- Anticipating L7, *Gray Atmosphere* gives solution for $T(\tau)$, so we'll need some reference λ for τ: normalize opacity to α(5000 A):

$$\alpha_\nu^{\mathrm{bf}}(5000A) = 5.5\times10^{-13}\,N_1\,\sigma_0\,(5000)^3$$

$$\begin{aligned}\alpha_\nu^{\mathrm{bf}}/\alpha_\nu^{\mathrm{bf}}(5000A) &= 149\,(\lambda/5000)^3 && \lambda \le 3646\,\mathrm{A} \\ &= (\lambda/5000)^3 && 3646\,\mathrm{A} \le \lambda \le 8206\,\mathrm{A} \\ &= 0.096\,(\lambda/5000)^3 && \lambda \ge 8206\,\mathrm{A}\end{aligned}$$



Have opacity, now need to solve ERT to get temperature structure and emergent intensity…

Formal Solution of ERT

The general equation of radiation transfer is:

$$\boxed{\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{\partial I_\nu}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial s} = \frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = j_\nu - \alpha_\nu I_\nu}$$

where s is the geometrical path along a ray and

$$\boxed{\frac{\partial I_\nu}{\partial s} = \frac{\partial I_\nu}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial I_\nu}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s}}$$

<u>Spherical Geometry:</u> Assuming steady state (time independence), adopting polar coordinates with $\mathrm{d}r = \cos\theta\, \mathrm{d}s$ and $r\mathrm{d}\theta = -\sin\theta\, \mathrm{d}s$ and taking a spherical star with azimuthially symmetric intensity we get:

$$\boxed{\frac{\partial I_\nu}{\partial s} = \cos\theta\frac{\partial I_\nu}{\partial r} + \frac{\sin\theta}{r}\frac{\partial I_\nu}{\partial \theta} = \mu\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial \mu} = j_\nu - \alpha_\nu I_\nu}$$

and with $S_\nu = j_\nu/\alpha_\nu$ we get: $$\frac{\mu}{\kappa_\nu \rho}\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{\kappa_\nu \rho\, r}\frac{\partial I_\nu}{\partial \mu} = S_\nu - I_\nu$$

This equation must be solved for stars with extended atmospheres. It is very difficult to solve and only recently have stellar atmosphere calculations been performed for this geometry. Most often it is assumed that stellar atmospheres are thin compared to their radius, so that the plane parallel approximation holds.

<u>Plane Parallel Geometry:</u> This is the approximation that we will use for the remainder of the discussion of "traditional" radiation transfer theory. In this approximation $d\theta/dr = 0$ and, with the radial optical depth, $d\tau_\nu = -\kappa_\nu \rho dr$, we get

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

For Sun: scaleheight, $h \sim 150$ km,
radius $R \sim 10^5$ km

We will now spend some time on finding solutions to this equation…

ERT: Moment Equations

Due to the complexity of the ERT, approximations are made to allow us to simplify the equations and find semi-analytic solutions. If we assume the source function, S_ν, is isotropic we may form various moment equations. First, perform angle averaging and apply

$$\frac{1}{4\pi}\int d\Omega = \frac{1}{2}\int d\mu$$

$$\frac{1}{2}\int_{-1}^{1} \mu \frac{dI_\nu}{d\tau_\nu} d\mu = \frac{1}{2}\int_{-1}^{1} I_\nu d\mu - \frac{1}{2}\int_{-1}^{1} S_\nu d\mu$$

$$\frac{dH_\nu(\tau_\nu)}{d\tau_\nu} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

$$-\frac{dH_\nu(z)}{dz} = \kappa_\nu \rho\, J_\nu(z) - \kappa_\nu \rho\, S_\nu(z)$$

If we multiply both sides by μ before applying the angle averaging we get:

$$\frac{1}{2}\int_{-1}^{1}\mu^2\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}\mathrm{d}\mu = \frac{1}{2}\int_{-1}^{1}\mu\, I_\nu \mathrm{d}\mu - \frac{1}{2}\int_{-1}^{1}\mu\, S_\nu \mathrm{d}\mu$$

0 for isotropic S_ν

$$\frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = H_\nu(\tau_\nu)$$

If we substitute this into the previous moment equation we get

$$\frac{\mathrm{d}^2K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

Formal Solution of ERT

$$\mu\frac{\mathrm{d}I_\nu(\tau_\nu,\mu)}{\mathrm{d}\tau_\nu} - I_\nu(\tau_\nu,\mu) = -S_\nu(\tau_\nu)$$

S_ν isotropic

Multiply by integrating factor, exp(-τ/μ) (ν-dependence implied):

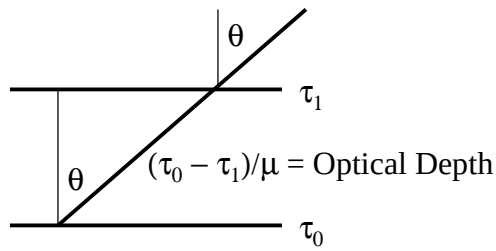
$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\left\{\mu\,\mathrm{e}^{-\tau/\mu}I_\nu(\tau,\mu)\right\} = -S_\nu(\tau)\,\mathrm{e}^{-\tau/\mu}$$

$$\left[\mu\,\mathrm{e}^{-\tau/\mu}I_\nu(\tau,\mu)\right]_{\tau_0}^{\tau_1} = -\int_{\tau_0}^{\tau_1} S_\nu(\tau)\,\mathrm{e}^{-\tau/\mu}\,d\tau$$

$$\mu\,\mathrm{e}^{-\tau_1/\mu}I_\nu(\tau_1,\mu) - \mu\,\mathrm{e}^{-\tau_0/\mu}I_\nu(\tau_0,\mu) = -\int_{\tau_0}^{\tau_1} S_\nu(\tau)\,\mathrm{e}^{-\tau/\mu}\,d\tau$$

$$I_\nu(\tau_1,\mu) = \mathrm{e}^{-(\tau_0-\tau_1)/\mu}I_\nu(\tau_0,\mu) - \frac{1}{\mu}\int_{\tau_0}^{\tau_1} S_\nu(\tau)\,\mathrm{e}^{-(\tau-\tau_1)/\mu}\,d\tau$$

$$= \mathrm{e}^{-(\tau_0-\tau_1)/\mu}I_\nu(\tau_0,\mu) + \frac{1}{\mu}\int_{\tau_1}^{\tau_0} S_\nu(\tau)\,\mathrm{e}^{-(\tau-\tau_1)/\mu}\,d\tau$$



Split into two regimes: Outward ($\mu > 0$), Inward ($\mu < 0$). Boundary conditions: $\tau_0 \rightarrow$ infty. No inward illumination, $I_\nu(\mu < 0) = 0$

$$\mu > 0: \quad I_\nu(\tau_1, \mu) = \frac{1}{\mu} \int_{\tau_1}^{\infty} S_\nu(\tau)\ \mathrm{e}^{-(\tau-\tau_1)/\mu}\ d\tau$$

$$\mu < 0: \quad I_\nu(\tau_1, \mu) = -\frac{1}{\mu} \int_{0}^{\tau_1} S_\nu(\tau)\ \mathrm{e}^{-(\tau-\tau_1)/\mu}\ d\tau$$

Intensity measures the source function weighted by $\exp(-\tau/\mu)$ along the beam up to the point of interest.

Moment Equations, Exponential Integrals, Operators

$$\int_{-1}^{1} I_\nu(\tau,\mu)\,\mu^n\,\mathrm{d}\mu = \int_0^1 \mu^n\,\mathrm{d}\mu \int_\tau^\infty S_\nu(\tau)\ \mathrm{e}^{-(\tau-\tau_1)/\mu}\ \frac{\mathrm{d}\tau}{\mu} + \int_{-1}^{0} \mu^n\,\mathrm{d}\mu \int_\tau^\infty S_\nu(\tau)\ \mathrm{e}^{-(\tau-\tau_1)/-\mu}\ \frac{\mathrm{d}\tau}{\mu}$$
$$= \int_\tau^\infty S_\nu(\tau)\ E_{n+1}(t-\tau)\,\mathrm{d}\tau + (-1)^n \int_0^\tau S_\nu(\tau)\ E_{n+1}(t-\tau)\,\mathrm{d}\tau$$

The exponential integrals E_n are defined by (Rutten 4.1.2)

$$E_n(x) \equiv \int_1^\infty \frac{\mathrm{e}^{-xw}}{w^n}\,\mathrm{d}w = \int_0^1 \mathrm{e}^{-x/\mu}\,\mu^{n-1}\,\frac{\mathrm{d}\mu}{\mu}$$

Tabulated in textbooks. For this course, we'll need approximations at small τ, so use

$$E_n(0) = \frac{1}{n-1}$$

Schwarzschild-Milne Equations

The Schwarzschild equation for the mean intensity:

$$\begin{aligned} J_\nu(\tau) &\equiv \frac{1}{2}\int_{-1}^{1} I_\nu(\tau,\mu)\,\mathrm{d}\mu \\ &= \frac{1}{2}\int_{\tau}^{\infty} S_\nu(\tau)\,E_1(t-\tau)\,\mathrm{d}t + \frac{1}{2}\int_{0}^{\tau} S_\nu(\tau)\,E_1(\tau-t)\,\mathrm{d}t \\ &= \frac{1}{2}\int_{0}^{\infty} S_\nu(\tau)\,E_1(|\,t-\tau\,|)\,\mathrm{d}t \end{aligned}$$

The Milne equation for the flux:

$$\begin{aligned} \mathcal{F}_\nu(\tau_\nu) &= \mathcal{F}_\nu^{+}(\tau_\nu) + \mathcal{F}_\nu^{-}(\tau_\nu) \\ &= 2\pi\int_{0}^{1} I_\nu(\tau_\nu)\,\mu\,\mathrm{d}\mu - 2\pi\int_{0}^{-1} I_\nu(\tau_\nu)\,\mu\,\mathrm{d}\mu \\ &= 2\pi\int_{0}^{1} S_\nu(t_\nu)\,E_2(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu - 2\pi\int_{0}^{1} S_\nu(t_\nu)\,E_2(\tau_\nu-t_\nu)\,\mathrm{d}t_\nu \end{aligned}$$

Completing the intensity moments in terms of exponential integrals, we get for the K_ν integral:

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_{0}^{\infty} S_\nu(t_\nu)\,E_3(|\,t_\nu-\tau_\nu\,|)\,\mathrm{d}t_\nu$$

Surface Values

The emergent intensity and flux at the stellar surface are:

$$I_\nu^{+}(0,\mu) = \int_{0}^{\infty} S_\nu(t_\nu)\,\mathrm{e}^{-\tau/\mu}\,\mathrm{d}\tau_\nu\,/\,\mu$$

$$\mathcal{F}_\nu^{+}(0,\mu) = 2\pi\int_{0}^{\infty} S_\nu(t_\nu)\,E_2(\tau_\nu)\,\mathrm{d}\tau_\nu$$

Operators

We can write the above equations in terms of *operators*. For the specific intensity we use the *Laplace Transform*:

$$\boxed{\mathcal{L}_{1/\mu}\{S_\nu(\tau_\nu)\} \equiv \int_0^\infty S_\nu(t_\nu)\,\mathrm{e}^{-\tau/\mu}\,\mathrm{d}\tau_\nu / \mu = I_\nu^+(0,\mu)}$$

In stellar atmospheres theory an important operator is the classical Lambda Operator, Λ_τ, defined by the RHS of the Schwarzschild eqn:

$$\boxed{\Lambda_\tau\{f(\tau)\} \equiv \frac{1}{2}\int_0^\infty f(\tau)\,E_1(|\,t-\tau\,|)\,\mathrm{d}t = J_\tau}$$

The Φ and χ operators are:

$$\boxed{\Phi_\tau\{S_\nu(t_\nu)\} \equiv 2\int_{\tau_\nu}^\infty S_\nu(t_\nu)\,E_2(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu - 2\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_2(\tau_\nu-t_\nu)\,\mathrm{d}t_\nu = F_\nu(\tau_\nu)}$$

$$\boxed{\chi_\tau\{S_\nu(t_\nu)\} \equiv 2\int_0^\infty S_\nu(t_\nu)\,E_3(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu = 4K_\nu(\tau_\nu)}$$