5. Formal Solution of ERT

- Ingredients to produce theoretical spectrum
- Rutten: 4.1
- Formal solution of plane-parallel ERT
- Exponential integrals and operators

Producing a Theoretical Spectrum

- Pure hydrogen atmosphere, $T_{\text{eff}} = 5600$ K
- Assume opacity independent of depth
- Determine wavelength dependent opacity
- Determine temperature structure
- Determine emergent intensity in optical range $3000$ Å < $\lambda$ < $8600$ Å
HI Continuous Extinction

For $912 < \lambda < 3646$ can ionize out of $n > 1$: need $N_2, N_3, N_4, \ldots$
For $3646 < \lambda < 8206$ can ionize out of $n > 2$: need $N_3, N_4, \ldots$
For $\lambda > 8206$ can ionize out of $n > 3$: need $N_4, \ldots$

Hydrogen Bound-Free Opacity

- Level populations for $T_{\text{eff}} = 5600$ K
- Use Boltzmann equation to get populations of H levels $n = 2, 3, \text{and } 4$
- Use Kramer’s opacity formula to get wavelength dependence
Calculate Level Populations

Use Boltzmann distribution, \( g_n = n^2 \):

\[
\frac{N_n}{N_1} = \frac{g_n}{g_1} \exp\left[-\frac{(\chi - \chi_n)}{kT}\right] = n^2 \exp\left[-\frac{\Delta E_{1n}}{kT}\right] = n^2 10^{-\frac{\Delta E_{1n}(\text{eV})5040/T}{\text{eV}}}\\
\]

Use Rydberg formula: \( \Delta E_{1n} = \chi \left(1 - \frac{1}{n^2}\right) \), \( \chi = 13.6 \text{ eV} \)

\[
\begin{align*}
N_2 / N_1 &= 4 \times 10^{\left(-10.2\times0.9\right)} = 2.6 \times 10^{-9} \\
N_3 / N_1 &= 9 \times 10^{\left(-11.9\times0.9\right)} = 1.2 \times 10^{-10} \\
N_4 / N_1 &= 16 \times 10^{\left(-12.75\times0.9\right)} = 5.4 \times 10^{-11}
\end{align*}
\]

Calculate Opacity

From L3, Kramer’s opacity for hydrogen b-f cross-section (\( m^2 \))

\[
\sigma_{\nu}^\text{bf} = 2.815 \times 10^{25} \frac{Z^4}{n^3 \nu^3} g_{\text{bf}} \quad \text{for } \nu \geq \nu_0 \\
= 1.044 \times 10^{-30} \frac{\lambda^5}{n^5} = \sigma_0 \frac{\lambda^3}{n^5}
\]

With \( \lambda \) in A. Total opacity = sum of absorption coefficients from all levels that can be ionized by photon at given \( \lambda \), times population of level

\[
\alpha_{\nu}^\text{bf} = N_2 \sigma_2 + N_3 \sigma_3 + N_4 \sigma_4 \quad \text{for } \lambda \leq 3647 \text{A} \\
= N_3 \sigma_3 + N_4 \sigma_4 \quad \text{for } 3648 \text{A} \leq \lambda \leq 8206 \text{A} \\
= N_4 \sigma_4 \quad \text{for } \lambda \geq 8207 \text{A}
\]

Ignore contributions from levels \( n > 4 \)
Calculate Opacity

For $\lambda < 3646$ Å there’s enough energy to ionize out of $n > 1$:

$$\alpha_{e}^{bf} = N_2 \sigma_2 + N_3 \sigma_3 + N_4 \sigma_4$$
$$= N_1 \sigma_0 \lambda^3 (1/32 N_2 / N_1 + 1/243 N_3 / N_1 + 1/1024 N_4 / N_1)$$
$$= 8.2 \times 10^{-11} N_1 \sigma_0 \lambda^3$$

For $3647 \text{ Å} < \lambda < 8206$ Å, can ionize out of $n > 2$

$$\alpha_{e}^{bf} = N_1 \sigma_0 \lambda^3 (1/243 N_3 / N_1 + 1/1024 N_4 / N_1)$$
$$= 5.5 \times 10^{-13} N_1 \sigma_0 \lambda^3$$

For $\lambda > 8207$ Å, can ionize out of $n > 3$

$$\alpha_{e}^{bf} = N_1 \sigma_0 \lambda^3 (1/1024 N_4 / N_1) = 5.3 \times 10^{-14} N_1 \sigma_0 \lambda^3$$

Calculate Opacity

- Anticipating L7, *Gray Atmosphere* gives solution for $T(\tau)$, so we’ll need some reference $\lambda$ for $\tau$: normalize opacity to $\alpha(5000$ Å):

$$\alpha_{e}^{bf} (5000Å) = 5.5 \times 10^{-13} N_1 \sigma_0 (5000)$$

$$\frac{\alpha_{e}^{bf}}{\alpha_{e}^{bf} (5000Å)} = 149 (\lambda / 5000)^3 \quad \lambda \leq 3646 \text{ Å}$$
$$= (\lambda / 5000)^3 \quad 3646 \text{ Å} \leq \lambda \leq 8206 \text{ Å}$$
$$= 0.096 (\lambda / 5000)^3 \quad \lambda \geq 8206 \text{ Å}$$
Formal Solution of ERT

The general equation of radiation transfer is:

$$\frac{dI_v}{ds} = \frac{dI_v}{dt} \frac{dt}{ds} + \frac{\partial I_v}{\partial s} = \frac{1}{c} \frac{\partial I_v}{\partial t} + \frac{\partial I_v}{\partial s} = j_v - \alpha_v I_v$$

where $s$ is the geometrical path along a ray and

$$\frac{\partial I_v}{\partial s} = \frac{\partial I_v}{\partial x} \frac{dx}{ds} + \frac{\partial I_v}{\partial y} \frac{dy}{ds} + \frac{\partial I_v}{\partial z} \frac{dz}{ds}$$

**Spherical Geometry:** Assuming steady state (time independence), adopting polar coordinates with $dr = \cos \theta \, ds$ and $r d\theta = -\sin \theta \, ds$ and taking a spherical star with azimuthally symmetric intensity we get:

$$\frac{\partial I_v}{\partial s} = \cos \theta \frac{\partial I_v}{\partial r} + \sin \theta \frac{\partial I_v}{\partial r} \frac{\partial r}{\partial \theta} + \mu \frac{\partial I_v}{\partial r} + \mu^2 \frac{\partial I_v}{\partial \mu} = j_v - \alpha_v I_v$$
and with \( S_{\nu} = j_{\nu}/\alpha_{\nu} \) we get:

\[
\mu \frac{\partial I_{\nu}}{\partial r} + \frac{1 - \mu^2}{\kappa_{\nu} \rho} \frac{\partial I_{\nu}}{\partial \mu} = S_{\nu} - I_{\nu}
\]

This equation must be solved for stars with extended atmospheres. It is very difficult to solve and only recently have stellar atmosphere calculations been performed for this geometry. Most often it is assumed that stellar atmospheres are thin compared to their radius, so that the plane parallel approximation holds.

**Plane Parallel Geometry:** This is the approximation that we will use for the remainder of the discussion of “traditional” radiation transfer theory. In this approximation \( d\theta/dr = 0 \) and, with the radial optical depth, \( d\tau_{\nu} = -\kappa_{\nu} \rho dr \), we get

\[
\mu \frac{dI_{\nu}}{d\tau_{\nu}} = I_{\nu} - S_{\nu}
\]

For Sun: scale height, \( h \sim 150 \text{ km} \),
radius \( R \sim 10^5 \text{ km} \)

We will now spend some time on finding solutions to this equation…

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**ERT: Moment Equations**

Due to the complexity of the ERT, approximations are made to allow us to simplify the equations and find semi-analytic solutions. If we assume the source function, \( S_{\nu} \), is isotropic we may form various moment equations. First, perform angle averaging and apply

\[
\frac{1}{4\pi} \int d\Omega = \frac{1}{2} \int d\mu
\]

\[
\frac{1}{2} \int_{-1}^{1} \mu \frac{dI_{\nu}}{d\tau_{\nu}} d\mu = \frac{1}{2} \int_{-1}^{1} I_{\nu} d\mu - \frac{1}{2} \int_{-1}^{1} S_{\nu} d\mu
\]

\[
\frac{dH_{\nu}(\tau_{\nu})}{d\tau_{\nu}} = J_{\nu}(\tau_{\nu}) - S_{\nu}(\tau_{\nu})
\]

\[
\frac{dH_{\nu}(z)}{dz} = \kappa_{\nu} \rho J_{\nu}(z) - \kappa_{\nu} \rho S_{\nu}(z)
\]
If we multiply both sides by \( \mu \) before applying the angle averaging we get:

\[
\frac{1}{2} \int_{-1}^{1} \mu^2 \frac{dI_v}{d\tau_v} d\mu = \frac{1}{2} \int_{-1}^{1} \mu I_v d\mu - \frac{1}{2} \int_{-1}^{1} \mu S_v d\mu
\]

\[
\frac{dK_v(\tau_v)}{d\tau_v} = H_v(\tau_v)
\]

0 for isotropic \( S_v \)

If we substitute this into the previous moment equation we get

\[
\frac{d^2K_v(\tau_v)}{d\tau_v^2} = J_v(\tau_v) - S_v(\tau_v)
\]

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**Formal Solution of ERT**

\[
\mu \frac{dI_v(\tau_v, \mu)}{d\tau_v} - I_v(\tau_v, \mu) = -S_v(\tau_v) \quad S_v \text{ isotropic}
\]

Multiply by integrating factor, \( \exp(-\tau/\mu) \) (\( \nu \)-dependence implied):

\[
\frac{d}{d\tau_v} \left[ \mu e^{-\tau/\mu} I_v(\tau, \mu) \right] = -S_v(\tau) \ e^{-\tau/\mu}
\]

\[
\left[ \mu e^{-\tau/\mu} I_v(\tau, \mu) \right]_{\tau_0}^{\tau_1} = -\int_{\tau_0}^{\tau_1} S_v(\tau) \ e^{-\tau/\mu} \ d\tau
\]

\[
\mu e^{-\tau_1/\mu} I_v(\tau_1, \mu) - \mu e^{-\tau_0/\mu} I_v(\tau_0, \mu) = -\int_{\tau_0}^{\tau_1} S_v(\tau) \ e^{-\tau/\mu} \ d\tau
\]

\[
I_v(\tau_1, \mu) = e^{-(\tau_0-\tau)/\mu} I_v(\tau_0, \mu) - \frac{1}{\mu} \int_{\tau_0}^{\tau_1} S_v(\tau) \ e^{-(\tau-\tau_1)/\mu} \ d\tau
\]

\[
= e^{-(\tau_0-\tau)/\mu} I_v(\tau_0, \mu) + \frac{1}{\mu} \int_{\tau_1}^{\tau_0} S_v(\tau) \ e^{-(\tau-\tau_0)/\mu} \ d\tau
\]
\[(\tau_0 - \tau_1)/\mu = \text{Optical Depth}\]

Split into two regimes: Outward \((\mu > 0)\), Inward \((\mu < 0)\). Boundary conditions: \(\tau_0 \rightarrow \text{infty}\). No inward illumination, \(I_\nu(\mu < 0) = 0\)

\[
\begin{align*}
\mu > 0: & \quad I_\nu(\tau, \mu) = \frac{1}{\mu} \int_0^\tau S_\nu(\tau) \ e^{-e^{(\tau-\tau_1)/\mu}} \ d\tau \\
\mu < 0: & \quad I_\nu(\tau, \mu) = -\frac{1}{\mu} \int_0^\tau S_\nu(\tau) \ e^{-e^{(\tau-\tau_1)/\mu}} \ d\tau
\end{align*}
\]

Intensity measures the source function weighted by \(\exp(-\tau/\mu)\) along the beam up to the point of interest.

\[
\int I_\nu(\tau, \mu) \mu^n \ d\mu = \int_0^\tau \mu^n \ d\mu \int_\tau S_\nu(\tau) \ e^{-e^{(\tau-\tau_1)/\mu}} \ d\tau + \int_0^\tau \mu^n \ d\mu \int_\tau S_\nu(\tau) \ e^{-e^{(\tau-\tau_1)/\mu}} \ d\tau
\]

\[
= \int_\tau S_\nu(\tau) \ E_{n+1}(t - \tau) \ d\tau + (-1)^n \int_0^\tau S_\nu(\tau) \ E_{n+1}(t - \tau) \ d\tau
\]

The exponential integrals \(E_n\) are defined by (Rutten 4.1.2)

\[
E_n(x) \equiv \int_1^\infty e^{-xw} \ dw = \int_0^1 e^{-x/\mu} \mu^{n-1} \ d\mu/\mu
\]

Tabulated in textbooks. For this course, we’ll need approximations at small \(\tau\), so use

\[
E_n(0) = \frac{1}{n-1}
\]
Schwarzschild-Milne Equations

The Schwarzschild equation for the mean intensity:

\[
J_\nu(\tau) = \frac{1}{2} \int_0^\infty I_\nu(\tau, \mu) \, d\mu
\]

\[
= \frac{1}{2} \int_0^\infty S_\nu(\tau) E_1(\tau - t) \, dt + \frac{1}{2} \int_0^\infty S_\nu(\tau) E_1(\tau - t) \, dt
\]

\[
= \frac{1}{2} \int_0^\infty S_\nu(\tau) E_1(|\tau - t|) \, dt
\]

The Milne equation for the flux:

\[
\mathcal{F}_\nu(\tau_\nu) = \mathcal{F}_\nu^+(\tau_\nu) + \mathcal{F}_\nu^-(\tau_\nu)
\]

\[
= 2\pi \int_0^{1/2} I_\nu(\tau_\nu, \mu) \, d\mu - 2\pi \int_0^{1/2} I_\nu(\tau_\nu, \mu) \, d\mu
\]

\[
= 2\pi \int_0^{1/2} S_\nu(t_\nu) E_1(t_\nu - \tau_\nu) \, dt_\nu - 2\pi \int_0^{1/2} S_\nu(t_\nu) E_1(t_\nu - \tau_\nu) \, dt_\nu
\]

Completing the intensity moments in terms of exponential integrals, we get for the \(K_\nu\) integral:

\[
K_\nu(\tau_\nu) = \frac{1}{2} \int_0^\infty S_\nu(t_\nu) E_1(|t_\nu - \tau_\nu|) \, dt_\nu
\]

Surface Values

The emergent intensity and flux at the stellar surface are:

\[
I_\nu^+(0, \mu) = \int_0^\infty S_\nu(t_\nu) e^{-\tau_\nu} \, d\tau_\nu / \mu
\]

\[
\mathcal{F}_\nu^+(0, \mu) = 2\pi \int_0^{1/2} S_\nu(t_\nu) E_1(t_\nu) \, dt_\nu
\]
Operators

We can write the above equations in terms of operators. For the specific intensity we use the Laplace Transform:

\[ \mathcal{L}_{\mu \mu} \{ S_\nu(\tau_\nu) \} \equiv \int_0^\infty S_\nu(t_\nu) e^{-\mu \tau_\nu} \, \tau_\nu \, d\mu / \mu = I^*_\nu(0, \mu) \]

In stellar atmospheres theory an important operator is the classical Lambda Operator, \( \Lambda_\nu \), defined by the RHS of the Schwarzschild eqn:

\[ \Lambda_\nu \{ f(\tau) \} = \frac{1}{2} \int_0^\infty f(\tau) \, E_1(\{1 - \tau\}) \, d\tau = J_\nu \]

The \( \Phi \) and \( \chi \) operators are:

\[ \Phi_\nu \{ S_\nu(t_\nu) \} \equiv 2 \int_0^\infty S_\nu(t_\nu) \, E_2(t_\nu - \tau_\nu) \, dt_\nu - 2 \int_0^\infty S_\nu(t_\nu) \, E_2(\tau_\nu - t_\nu) \, dt_\nu = F_\nu(\tau_\nu) \]

\[ \chi_\nu \{ S_\nu(t_\nu) \} \equiv 2 \int_0^\infty S_\nu(t_\nu) \, E_3(t_\nu - \tau_\nu) \, dt_\nu = 4 K_\nu(\tau_\nu) \]