6. Analytic Solutions I

- Rutten: 4.1, 4.2
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- Rosseland Mean Opacity
- Eddington Approximation

Schwarzschild-Milne Equations

For a semi infinite atmosphere with no external illumination:

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(|t_\nu - \tau_\nu|)\, \mathrm{d}t_\nu$$

$$\mathcal{F}_\nu(\tau_\nu) = 2\pi\int_0^1 S_\nu(t_\nu)\, E_2(t_\nu - \tau_\nu)\, \mathrm{d}t_\nu - 2\pi\int_0^1 S_\nu(t_\nu)\, E_2(\tau_\nu - t_\nu)\, \mathrm{d}t_\nu$$

Surface Values

$$I_\nu^+(0,\mu) = \int_0^\infty S_\nu(\tau_\nu)\, \mathrm{e}^{-\tau_\nu/\mu}\, \mathrm{d}\tau_\nu / \mu$$

$$\mathcal{F}_\nu^+(0,\mu) = 2\pi\int_0^\infty S_\nu(\tau_\nu)\, E_2(\tau_\nu)\, \mathrm{d}\tau_\nu$$

Operators

We can write the above equations in terms of *operators*. For the specific intensity we use the *Laplace Transform*:

$$\mathcal{L}_{1/\mu}\{S_\nu(\tau_\nu)\} \equiv \int_0^\infty S_\nu(\tau_\nu)\, e^{-\tau_\nu/\mu}\, d\tau_\nu / \mu = I_\nu^+(0,\mu)$$

In stellar atmospheres theory an important operator is the classical Lambda Operator, Λ_τ, defined by the RHS of the Schwarzschild eqn:

$$\Lambda_\tau\{f(\tau)\} \equiv \frac{1}{2}\int_0^\infty f(\tau)\, E_1(|t-\tau|)\, dt$$

The Φ and χ operators are:

$$\Phi_\tau\{S_\nu(t_\nu)\} \equiv 2\int_{\tau_\nu}^\infty S_\nu(t_\nu)\, E_2(t_\nu-\tau_\nu)\, dt_\nu - 2\int_0^{\tau_\nu} S_\nu(t_\nu)\, E_2(\tau_\nu-t_\nu)\, dt_\nu = F_\nu(\tau_\nu)$$

$$\chi_\tau\{S_\nu(t_\nu)\} \equiv 2\int_0^\infty S_\nu(t_\nu)\, E_3(t_\nu-\tau_\nu)\, dt_\nu = 4K_\nu(\tau_\nu)$$

Some Properties

$$\Lambda_\tau\{1\} = 1 - \frac{1}{2}E_2(\tau)$$

$$\Lambda_\tau\{t\} = \tau - \frac{1}{2}E_3(\tau)$$

$$\Lambda_\tau\{t^2\} = \frac{2}{3} + \tau^2 - E_4(\tau)$$

Lambda operator applied to *S* gives *J*:

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_1(|t-\tau_\nu|)\, dt = \Lambda_{\tau_\nu}\{S_\nu(t_\nu)\}$$

Analytic Solutions

Optically thick radiation transfer only has analytic solutions at large depth where LTE holds and the radiation field is very nearly isotropic. In shallower, optically thinner layers, approximations are inevitable. The most important one is the (first) Eddington Approximation, which we will work up to below.

Approximations are based on power law expansions of $S(\tau)$

Recall Taylor-McLaurin series:

$$f(\tau) = \sum_{n=0}^{\infty} a_n (\tau - \tau_0)^n$$

$$a_n = \left. \frac{f^n(\tau)}{n!} \right|_{\tau=\tau_0}$$

Expand at surface: $\tau_0 = 0$
Interior: pick some τ_0

Approximations at the Surface

Eddington-Barbier Approximation: This approximation for the emergent specific intensity is based on the polynomial expansion:

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} a_n \tau_\nu^n$$

which produces, using the linearity of operators (Tutorial Exercise):

$$I_\nu^+(0,\mu) = \mathcal{L}_{1/\mu}\{S_\nu(\tau_\nu)\} = \sum_{n=0}^{\infty} n!\, a_n\, \mu^n$$

$$\begin{aligned} J_\nu(\tau_\nu) &= \Lambda_\nu\{S_\nu(\tau_\nu)\} \\ &= a_0\Lambda_\nu\{1\} + a_1\Lambda_\nu\{t\} + a_2\Lambda_\nu\{t^2\} + \ldots \\ &\approx a_0\left[1 - \frac{1}{2}E_2(\tau_\nu)\right] + a_1\left[\tau_\nu + \frac{1}{2}E_3(\tau_\nu)\right] + a_2\left[\frac{2}{3} + \tau_\nu^2 - E_4(\tau_\nu)\right] \\ &\approx a_0 + a_1\tau_\nu + a_2\tau_\nu^2 + \frac{2}{3}a_2 - \frac{a_0}{2}E_2(\tau_\nu) + \frac{a_1}{2}E_3(\tau_\nu) - a_2E_4(\tau_\nu) \end{aligned}$$

$$\begin{aligned} F_\nu(\tau_\nu) &= \Phi_\nu\{S_\nu(\tau_\nu)\} \\ &= 2a_0E_3(\tau_\nu) + a_1\left[\frac{4}{3} - 2E_4(\tau_\nu)\right] + a_2\left[\frac{8}{3}\tau_\nu + 4E_5(\tau_\nu)\right] + \ldots \end{aligned}$$

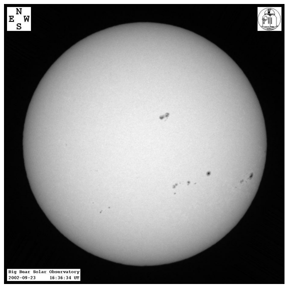
At the surface, the above are approximately given by:

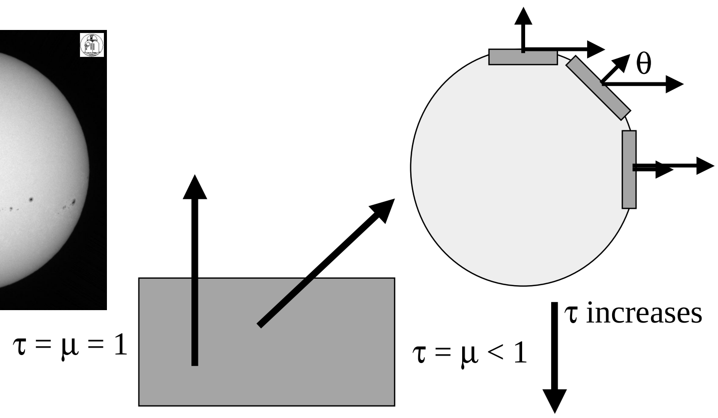
$$\begin{aligned} I_\nu^+(0,\mu) &\approx a_0 + a_1\mu = S_\nu(\tau_\nu = \mu) \\ J_\nu(0) &\approx a_0 + \frac{2a_2}{3} - \frac{a_0}{2} + \frac{a_1}{4} - \frac{a_2}{3} \\ &\approx \frac{a_0}{2} + \frac{a_1}{4} + \frac{a_2}{3} \approx \frac{1}{2}S_\nu(\tau_\nu = 1/2) \\ F_\nu(0) &= a_0 + \frac{2a_1}{3} + a_2 + \ldots \approx S_\nu(\tau_\nu = 2/3) \end{aligned}$$

$$\begin{aligned} I_\nu^+(0,\mu) &\approx S_\nu(\tau_\nu = \mu) \\ J_\nu(0) &\approx \frac{1}{2}S_\nu(\tau_\nu = 1/2) \\ F_\nu(0) &\approx S_\nu(\tau_\nu = 2/3) \end{aligned}$$

These are the ***Eddington-Barbier Approximations*** for the emergent intensity, mean intensity, and net flux at the surface in the absence of external illumination (i.e., $I_\nu^-(0)=0$). They are exact for a source function that is linear with optical depth $S_\nu(\tau_\nu) = a_0 + a_1\tau_\nu$ and are then easily found from the definitions of *I, J, F* using the boundary condition $I_\nu^-(0)=0$.

Limb Darkening





Towards limb, $\tau = \mu$ is at a shallower depth. If $S(\tau) = a + b\,\tau$ then…
$I = S\,(\tau = \mu) = a + b\,\mu$, so I (Limb) < I (Center)

Second Eddington Approximation: A homogeneous medium with $S_\nu = a_0$ is known as a *Lambert Radiator* having $I_\nu^+(0,\mu) = S_\nu = a_0$ for all outward directions $\mu > 0$. Also, $J_\nu(0) = S_\nu / 2 = a_0 / 2 = I_\nu(0) / 2$, $F_\nu(0) = S_\nu = I_\nu(0) = a_0$ and therefore $F_\nu(0) = 2\,J_\nu(0) = 4\,H_\nu(0)$ or $\mathcal{F}_\nu(0) = 2\,\pi\,J_\nu(0)$. The latter relation is called the *Second Eddington Approximation*. It follows directly from the definition of astrophysical flux by setting:

$$\boxed{\begin{aligned} F(0) &\equiv 2\int_{-1}^{+1} I_\nu(0,\mu)\,\mu\,\mathrm{d}\mu = 2\int_{0}^{+1} I_\nu(0,\mu)\,\mu\,\mathrm{d}\mu \\ &\approx 2\left\langle I_\nu^+(0,\mu)\right\rangle \int_0^1 \mu\,\mathrm{d}\mu \approx 2J_\nu(0) \end{aligned}}$$

using the absence of incident radiation and recognizing that $J_\nu(0) = 1/2 < I_\nu^+(0,\mu) >$ for the same reason. This approximation is exact for a Lambert radiator, simply expressing that $F_\nu(0) = F_\nu^+(0)$ Represents an average only over outward directions ($\mu > 0$), while $J_\nu(0)$ is an average over all μ. In general it is a very coarse approximation.

Approximations at Large Depth

At large depths ($\tau >> 1$) radiation transfer becomes simple because all scale lengths are larger than the photon mean free path. So photons are locally trapped in a nearly homogeneous environment even while randomly walking about via scattering. The radiation field therefore approaches isotropy. Also, the density is large enough that collisional photon destruction outweighs scattering. The conditions therefore approach TE, making LTE a valid assumption. Expanding S_ν in a Taylor-McLaurin series gives:

$$S_\nu(t_\nu) = \sum_{n=0}^{\infty} \frac{(t_\nu - \tau_\nu)^n}{n!} \left[\frac{\mathrm{d}^n S_\nu(t_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}$$

Substitute into equation for I at arbitrary interior point τ of a semi infinite atmosphere:

$$\mu > 0: \quad I_\nu(\tau,\mu) = \int_\tau^\infty S_\nu(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt/\mu$$

$$\mu < 0: \quad I_\nu(\tau,\mu) = -\int_0^\tau S_\nu(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt/\mu$$

Substituting this into the formal solution for I_ν and using

$$\int_{x_1}^{x_2} x^n \mathrm{e}^{ax}\,\mathrm{d}x = \left[\frac{x^n \mathrm{e}^{ax}}{a}\right]_{x_1}^{x_2} - \frac{n}{a}\int_{x_1}^{x_2} x^{n-1}\mathrm{e}^{ax}\,\mathrm{d}x$$

$$\int_{x_1}^{x_2} x^n \mathrm{e}^{-x}\,\mathrm{d}x = \left[-\mathrm{e}^{-x}\left\{x^n + nx^{n-1} + n(n-1)x^{n-2} + \ldots + n!\right\}\right]_{x_1}^{x_2}$$

$$\int_0^\infty x^n \mathrm{e}^{-x}\,\mathrm{d}x = n!$$

$$\begin{aligned}
I_\nu^+(\tau_\nu,\mu) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \int_{\tau_\nu}^{\infty} (t_\nu - \tau_\nu)^n\ \mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu \\
&= \sum_{n=0}^{\infty} \left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \frac{1}{n!}\int_0^\infty x^n\ \mathrm{e}^{-x/\mu}\,\mathrm{d}x/\mu \\
&= \sum_{n=0}^{\infty} \left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu} \frac{\mu^n}{n!}\int_0^\infty x^n\ \mathrm{e}^{-x/\mu}\,\mathrm{d}x/\mu \\
&= \sum_{n=0}^{\infty} \mu^n \left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}
\end{aligned}$$

$$
\begin{aligned}
I_\nu^-(\tau_\nu,\mu) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}\left[-\int_0^{\tau_\nu}(t_\nu-\tau_\nu)^n\,\mathrm{e}^{-(t_\nu-\tau_\nu)/\mu}\,\mathrm{d}t_\nu/\mu\right]\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}(-1)^n\left[\int_0^{\tau_\nu}(t_\nu-\tau_\nu)^n\,\mathrm{e}^{-(t_\nu-\tau_\nu)/|\mu|}\,\mathrm{d}t_\nu/|\mu|\right]\\
&= \sum_{n=0}^{\infty}\left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}(-1)^n\frac{|\mu|^n}{n!}\left[\int_0^{\tau_\nu/\mu}x^n\,\mathrm{e}^{-x|}\,\mathrm{d}x\right]\\
&= \sum_{n=0}^{\infty}\mu^n\left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}\left[1-\frac{\mathrm{e}^{-\tau_\nu/|\mu|}}{n!}\left\{(\tau_\nu/|\mu|)^n+n(\tau_\nu/|\mu|)^{n-1}+\ldots+n!\right\}\right]
\end{aligned}
$$

The term in brackets [1 - …] goes to 1 for large τ_ν. Thus

$$
I_\nu(\tau_\nu,\mu) = S_\nu(\tau_\nu)+\mu\left[\frac{\mathrm{d}S_\nu(\tau_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu}+\mu^2\left[\frac{\mathrm{d}^2 S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu}+\ldots
$$

This holds for all directions $-1<\mu<1$ when $\tau_\nu \gg 1$, and holds also for $\mu>0$ at smaller depth.

For $\tau_\nu \gg 1$ substituting the above in the formula for mean intensity:

$$
J_\nu(\tau_\nu)=\frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{\mathrm{d}^n S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^n}\right]_{\tau_\nu}\int_{-1}^{1}\mu^n\,\mathrm{d}\mu=\sum_{k=0}^{\infty}\frac{1}{2k+1}\left[\frac{\mathrm{d}^{2k} S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^{2k}}\right]_{\tau_\nu}
$$

Using similar expressions for F_ν and K_ν, we get

$$
\begin{aligned}
J_\nu(\tau_\nu) &= S_\nu(\tau_\nu)+\frac{1}{3}\left[\frac{\mathrm{d}^2 S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu}+\ldots\\
F_\nu(\tau_\nu) &= \frac{4}{3}\left[\frac{\mathrm{d}S_\nu(\tau_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu}+\frac{4}{5}\left[\frac{\mathrm{d}^3 S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^3}\right]_{\tau_\nu}+\ldots\\
K_\nu(\tau_\nu) &= \frac{1}{3}S_\nu(\tau_\nu)+\frac{1}{5}\left[\frac{\mathrm{d}^2 S_\nu(\tau_\nu)}{\mathrm{d}t_\nu^2}\right]_{\tau_\nu}+\ldots
\end{aligned}
$$

These expressions rapidly converge for $\tau_\nu \gg 1$, so at large depth we get:

$$I_\nu(\tau_\nu,\mu) \approx S_\nu(\tau_\nu) + \mu\left[\frac{\mathrm{d}S_\nu(\tau_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu}$$

$$J_\nu(\tau_\nu) \approx S_\nu(\tau_\nu)$$

$$F_\nu(\tau_\nu) \approx \frac{4}{3}\left[\frac{\mathrm{d}S_\nu(\tau_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu}$$

$$K_\nu(\tau_\nu) \approx \frac{1}{3}S_\nu(\tau_\nu)$$

Here the isotropic component of the radiation field J_ν is set by the value of the source function whereas the anisotropic component F_ν is determined by the gradient $\mathrm{d}S_\nu / \mathrm{d}\tau_\nu$.

Diffusion Approximation: *The* ***Rosseland*** *or* ***Diffusion Approximation*** holds sufficiently deep inside a star where I_ν is nearly isotropic and where LTE holds so that $S_\nu = B_\nu$.

$$I_\nu(\tau_\nu,\mu) \approx B_\nu(\tau_\nu) + \mu\left[\frac{\mathrm{d}B_\nu(t_\nu)}{\mathrm{d}t_\nu}\right]_{\tau_\nu}$$

$$J_\nu(z) \approx B_\nu(\tau_\nu)$$

$$\mathcal{F}_\nu(z) \approx 2\pi\int_{-1}^{+1}\mu\, I_\nu\,\mathrm{d}\mu \approx \frac{4\pi}{3}\frac{\mathrm{d}B_\nu(z)}{\mathrm{d}\tau_\nu}$$

The monochromatic flux is now expressed in the gradient of B_ν in optical depth. This equation has the general form of a diffusion process where the transported flux of a quantity equals the product of a diffusion coefficient and a spatial gradient in that quantity.

Rosseland Mean Extinction: In order to recast the diffusion approximation into the familiar expression for the total flux $\mathcal{F}$ as a function of the geometrical radial temperature gradient dT / dz, we use the ***Rosseland Mean Extinction coefficient***:

$$\frac{1}{\alpha_R} \equiv \frac{\int_0^\infty (1/\alpha_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu}{\int_0^\infty (\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu} \quad ; \quad \frac{1}{\kappa_R} \equiv \frac{\int_0^\infty (1/\kappa_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu}{\int_0^\infty (\mathrm{d}B_\nu/\mathrm{d}T)\,\mathrm{d}\nu}$$

Where $\kappa_R(z) = \alpha_R(z) / \rho(z)$. It averages the extinction similarly to the formula for combining parallel resistors: $1/R = 1/R_1 + 1/R_2 + \ldots$ The extinction represents resistance to the photon flux which favours the more transparent spectral windows. The Planck function temperature sensitivity, $\mathrm{d}B_\nu/\mathrm{d}T$ enters as a weighting function for the same reason. It produces larger flux from a given spatial temperature gradient at frequencies where it is large.

The total energy flow is now given by:

$$\begin{aligned}\mathcal{F}(z) &= \int_0^\infty \mathcal{F}_\nu(z)\,\mathrm{d}\nu \\ &\approx -\frac{4\pi}{3}\int_0^\infty \frac{1}{\alpha_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}z}\,\mathrm{d}\nu \approx -\frac{4\pi}{3}\int_0^\infty \frac{1}{\alpha_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}z}\,\mathrm{d}\nu \\ &\approx -\frac{16}{3}\frac{\sigma T^3}{\alpha_R}\frac{\mathrm{d}T}{\mathrm{d}z} \approx -\frac{1}{3}\frac{c}{\rho\kappa_R}\frac{\mathrm{d}u}{\mathrm{d}z}\end{aligned}$$

Where u is the total energy density defined before $u = (4\sigma / c)\, T^4$. This diffusion equation is also called the *radiation conduction equation*. It says a negative temperature gradient is required to let net radiative flux diffuse outwards through a star by thermal absorptions and re-emissions with a mean free photon path $l = 1/\rho\,\kappa_R$. In the solar interior l is only a few millimetres, making the optical depth from the surface $\tau_\nu \sim 10^{11}$, so the diffusion approximations are very accurate.

The Eddington Approximation

Using the equations for J and K at large depth we get the ***First Eddington Approximation***, often called ***the*** Eddington approximation

$$\boxed{K_\nu(\tau_\nu) \approx \frac{1}{3} J_\nu(\tau_\nu)}$$

Validity: It is exact for isotropic radiation. It is also exact, at any depth τ_ν, when $I_\nu(\tau,\mu)$ can be expanded in odd powers of μ, with all even coefficients $a_i = 0$ in

$$\boxed{I_\nu(\tau_\nu) = \sum_{i=0}^{n} a_i(\tau_\nu)\,\mu^i}$$

This implies the Eddington approximation may hold for $\tau_\nu < 1$, in contrast to the approximations at large depth and the diffusion equation which requires LTE and therefore only holds for $\tau_\nu > 1$.