7. Analytic Solutions II

- Rutten: 7.3
- Eddington two stream approximation
- Gray atmosphere
- Limb darkening
- Theoretical spectrum: temperature structure and emergent flux

Why Do Gray Atmosphere?

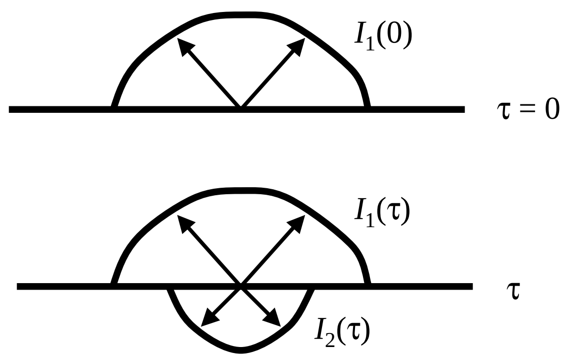
- Opacity independent of frequency
- Only true gray opacity is electron scattering
- Demonstrates radiative equilibrium
- Gives first guess at temperature structure
- Can relate to more general/realistic situations
- Can get exact solution: test approximate numerical techniques
- Can set exam questions…

Eddington Two Stream Approach

Crude representation of radiation field:

$$
\begin{aligned}
I(\tau,\mu) &= I_1(\tau) \quad \mu > 0 \\
&= I_2(\tau) \quad \mu > 0
\end{aligned}
$$

Where I_1 and I_2 are constant at any given τ
Surface boundary condition: $I_2(\tau = 0) = 0$ (no incident radiation)



Substitute I_1 and I_2 into integral expressions for J, H, K:

$$
\begin{aligned}
J(\tau) &= \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\,\mathrm{d}\mu = \frac{1}{2}\int_{0}^{1} I_1(\tau)\,\mathrm{d}\mu + \frac{1}{2}\int_{-1}^{0} I_2(\tau)\,\mathrm{d}\mu = \frac{1}{2}[I_1(\tau) + I_2(\tau)] \\
H(\tau) &= \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\mu\,\mathrm{d}\mu = \frac{1}{2}\int_{0}^{1} I_1(\tau)\mu\,\mathrm{d}\mu + \frac{1}{2}\int_{-1}^{0} I_2(\tau)\mu\,\mathrm{d}\mu = \frac{1}{4}[I_1(\tau) + I_2(\tau)] \\
K(\tau) &= \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\mu^2\,\mathrm{d}\mu = \frac{1}{2}\int_{0}^{1} I_1(\tau)\mu^2\,\mathrm{d}\mu + \frac{1}{2}\int_{-1}^{0} I_2(\tau)\mu^2\,\mathrm{d}\mu = \frac{1}{6}[I_1(\tau) + I_2(\tau)]
\end{aligned}
$$

The equations for J and K give the first Eddington approximation:

$$J(\tau) = 3K(\tau)$$

Surface condition $I_2(\tau = 0) = 0$ gives the second Eddington approx:

$$J(0) = I_1(0)/2,\ H(0) = I_1(0)/4 \Rightarrow J(0) = 2H(0)$$

The major assumption in the gray atmosphere is that the opacity is independent of frequency: $d\kappa_\nu / d\nu = 0$. The ERT simplifies considerably since we can integrate over frequency.

$$\boxed{\mu \frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} = I - S}$$

where

$$\boxed{I = \int_0^\infty I_\nu \,\mathrm{d}\nu \quad ; \quad S = \int_0^\infty S_\nu \,\mathrm{d}\nu}$$

We now assume:

1. Opacity independent of frequency (see above)
2. Radiative equilibrium: Total radiative flux is constant throughout atmosphere: $dF / d\tau = 0$
3. Atmosphere is in LTE: Hence $S_\nu(T) = B_\nu(T)$ and $S(T) = B(T)$

A useful relation is the frequency integrated Planck function:

$$\boxed{B(T) = \int_0^\infty B_\nu(T)\,\mathrm{d}\nu = \frac{\sigma T^4}{\pi}}$$

Recall the Moment equations from L6. Be able to derive these.

$$\boxed{\begin{aligned} \frac{\mathrm{d}H_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} &= J_\nu(\tau_\nu) - S_\nu(\tau_\nu) \\ \frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} &= H_\nu(\tau_\nu) \\ \frac{\mathrm{d}^2 K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} &= J_\nu(\tau_\nu) - S_\nu(\tau_\nu) \end{aligned}}$$

Using the frequency integrated terms, the moment equations for the ERT now become:

$$\boxed{\begin{aligned}\frac{\mathrm{d}H(\tau)}{\mathrm{d}\tau} &= J(\tau) - S(\tau)\\ \frac{\mathrm{d}K(\tau)}{\mathrm{d}\tau} &= H(\tau)\end{aligned}}$$

Radiative equilibrium implies the flux is constant throughout the atmosphere, so $\mathrm{d}H / \mathrm{d}\tau = 0$, since $H = F / 4$.

This then gives $S(\tau) = J(\tau)$ and $K(\tau) = H(\tau + q)$, since $H(\tau) = H$ (constant) and q is a constant of integration.

Using the Eddington approximation, $J(\tau) = 3K(\tau)$, we get

$$\boxed{J(\tau) = 3H(\tau + q)}$$

Using the second Eddington approximation, $J(\tau = 0) = 2H$, we get $q = 2/3$, which gives the Milne-Eddington approximation for a gray atmosphere:

$$\boxed{S(\tau) \approx \frac{3}{4}\left(\tau + \frac{2}{3}\right)F}$$

since $F = 4H$ and $S = J$. Here $F = B = (\sigma/\pi)T_{\mathrm{eff}}^4$.

The exact solution is usually written as:

$$\boxed{S(\tau) = \frac{3}{4}[\tau + q(\tau)]F}$$

where $q(\tau)$ is the *Hopf function* which varies slowly with τ.

Temperature Structure: Assuming LTE gives $S(\tau) = B(\tau) = (\sigma/\pi)T^4$ and using the Milne-Eddington approx above, we get

$$\boxed{T(\tau) \approx \left(\frac{3}{4}\tau + \frac{1}{2}\right)^{1/4} T_{\text{eff}}}$$

In the above we get $T_{\text{eff}} = T(\tau = 2/3)$ as it should

Limb Darkening: Using the Eddington-Barbier surface relation $I(0,\mu) = S(\tau = \mu)$ gives $I(0,\mu) = 3/4F(\mu + 2/3)$, so the centre-to-limb variation for a gray star is:

$$\boxed{\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}(\mu + 2/3)}$$

This gives $I(0,0) / I(0,1) = 0.4$, which is in excellent agreement with Solar observed limb darkening

Theoretical Spectrum

- Calculate wavelength dependent opacity: hydrogen bound-free opacity
- Temperature structure: Gray atmosphere:
  $T^4 = T^4_{\text{eff}}(3\tau/4+1/2)$
- Emergent flux: LTE + Eddington-Barbier:
  $F = S(\tau = 2/3) = B(T[\tau = 2/3])$

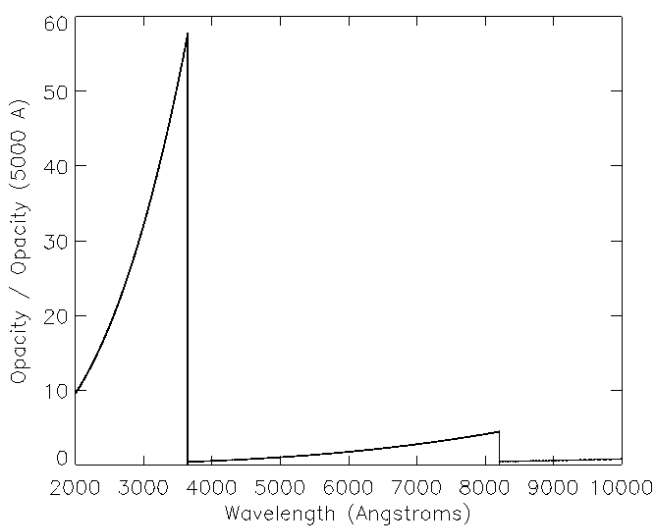
Temperature Structure

Assume gray temperature structure:Opacity isn't really gray, so let's pick 5000A as a representative wavelength: $\tau_{5000} = \tau(\lambda = 5000\text{ A})$, and assume temperature structure is

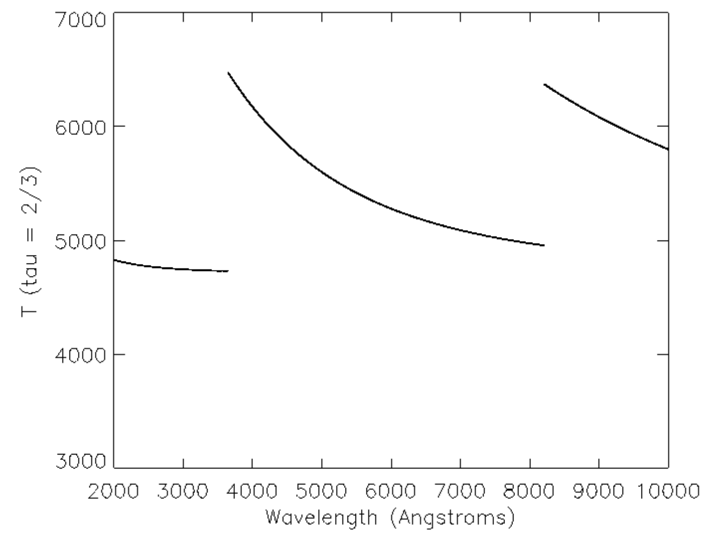
$$\boxed{T^4(\tau) = T_{\rm eff}^4\left(\frac{3}{4}\tau_{5000} + \frac{1}{2}\right)}$$

Eddington-Barbier gives $F(\lambda) = S(\tau_\lambda = 2/3) = B(T[\tau_\lambda = 2/3])$
Radiation comes from $\tau_\lambda = 2/3$, so want temperature at this depth
If $\tau_\lambda = 2/3$ then $\tau_{5000} = \tau_\lambda /(\kappa_\lambda / \kappa_{5000})$
So temperature at $\tau_\lambda = 2/3$ is

$$\boxed{T^4(\tau_\lambda) = T_{\rm eff}^4\left(\frac{3}{4}\times\frac{2}{3}[\kappa_\lambda / \kappa_{5000}] + \frac{1}{2}\right)}$$



Have opacity: $\kappa_\lambda / \kappa_{5000}$, from L6, so can get temperature structure…



Have temperature structure, so can now get emergent flux…

Emergent Flux

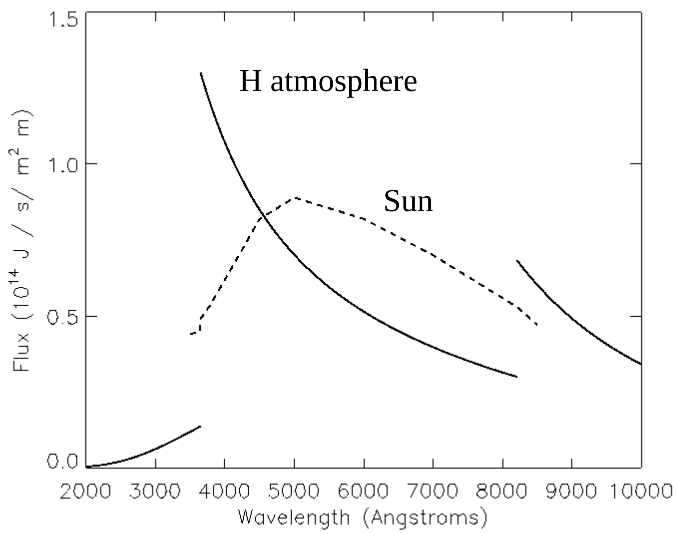
From Eddington-Barbier surface relation:

$$\boxed{F_\lambda(0) = \pi\, B_\lambda(T[\tau_\lambda = 2/3])}$$

Remember $\nu B_\nu = \lambda B_\lambda$ so

$$\boxed{F_\lambda(0) = \frac{2\pi h c^2}{\lambda^5}\frac{1}{\exp(hc/\lambda k T[\tau_\lambda]) - 1}}$$

Where we use the $T(\tau_\lambda)$ relation derived above to finally give the emergent flux spectrum…



Flux from pure hydrogen atmosphere is different from Sun
Solar opacity dominated by H^- opacity