1.1 Motivation

- Observational: How do we explain stellar properties as demonstrated, e.g. by the H-R diagram?
- Theoretical: How does an isolated, self-gravitating body of gas behave?
- Aims:
  - Identify and understand physical processes
  - Make simple models
  - Understand fundamental properties of stars
  - Learn how stars form and evolve

1.2 The H-R diagram

- Solar neighbourhood
  - stars of known distance.
  - Main sequence, giants, supergiants, white dwarfs
  - All masses and evolutionary stages present
Cluster H-R diagrams

- Open and globular clusters
  - “Snapshots” at a single age
  - Assume all stars formed at same time
  - All stars at same distance
  - Open clusters: disc population, young, solar-like metallicity
  - Globulars: Halo pop, old, metal-poor

Typical globular-cluster HR diagram.

1.3 Temperature-luminosity relation

- Most important empirical relation between stellar properties.
- Theoretical H-R diagram: plot
  
  \[ \log \left( \frac{L}{L_{\text{Sun}}} \right) \text{ against } \log \left( \frac{T_{\text{eff}}}{T_{\text{eff, Sun}}} \right) \]

- Main sequence shows luminosity correlated with effective temperature:
  
  \[ \frac{L}{L_{\text{Sun}}} = \left( \frac{T_{\text{eff}}}{T_{\text{eff, Sun}}} \right)^{\alpha} \]
  
  where on average \( \alpha = 6.67 \).
1.4 Mass - luminosity relation

- From masses of many binary stars, find for m-s stars:
  \[
  \frac{L}{L_{\text{Sun}}} = \left( \frac{M}{M_{\text{Sun}}} \right)^{4.0\pm0.02} \quad \text{for} \quad 0.4 < M < 10M_{\text{Sun}}
  \]
  \[
  \frac{L}{L_{\text{Sun}}} = \left( \frac{M}{M_{\text{Sun}}} \right)^{3.6\pm0.1} \quad \text{for} \quad 5 \leq M \leq 40M_{\text{Sun}}
  \]
- Theory of stellar structure must reproduce this!

1.6 Definitions

- Symbols & units commonly used for stellar parameters:
  - Stellar mass (M) -- solar units M/M_{\text{Sun}}
  - Stellar radius (R) -- solar units R/R_{\text{Sun}}
  - Surface gravity (g=GM/R^2) -- ms^{-2}
  - Effective temperature (T_{\text{eff}}) -- K
  - Stellar luminosity (L) -- solar units L/L_{\text{Sun}}
  - Composition (X, Y, Z) -- mass fractions of H (X), He (Y) and other elements (Z).
  - Stellar age (t) -- years
Example: The Sun

\[ M = 1 M_{\text{Sun}} = 1.99 \times 10^{30} \text{ kg} \]
\[ R = 1 R_{\text{Sun}} = 6.96 \times 10^8 \text{ m} \]
\[ g = 2.74 \times 10^2 \text{ m s}^{-2} \]
\[ T_{\text{eff}} = 5780 \text{ K} \]
\[ L = 1 L_{\text{Sun}} = 3.86 \times 10^{26} \text{ W} \]
\[ X = 0.71, \ Y = 0.265, \ Z = 0.025 \]
\[ t \cong 4.6 \times 10^9 \text{ y} \]

1.7a Dynamical (free-fall) timescale

- Ball of gas supported by pressure.
- How long to collapse if support removed?
- Spherical shell, initially at rest with radius \( R_0 \) encloses mass \( M \).
- After collapse to radius \( r \), inward velocity conserves PE + KE:

\[
\frac{1}{2} \left( \frac{dr}{dt} \right)^2 = \frac{GM}{r} - \frac{GM}{R_0}
\]

- Hence

\[
t_{\text{ff}} = \int_{R_0}^{0} \frac{dr}{\left( \frac{2GM}{r} - \frac{2GM}{R_0} \right)^{1/2}}
\]

Choose negative root to ensure positive answer!
Example: Solar free-fall time

Define $x = r / R_0$:

$$t_{ff} = \left( \frac{R_0^3}{2GM} \right)^{1/2} \int_0^1 \left( \frac{x}{1-x} \right)^{1/2} dx = \frac{\pi}{2} \left( \frac{R_0^3}{2GM} \right)^{1/2} ,$$

(use substitution $x = \sin^2 \theta$ to evaluate integral).

- cf. period of particle orbiting at $R_0$.
- Numerical value:

$$t_{ff} = 1770 \left( \frac{R_0}{R_{\text{Sun}}} \right)^{3/2} \left( \frac{M_{\text{Sun}}}{M} \right)^{1/2} \text{ s.}$$

1.7b Thermal (Kelvin) timescale

- Treat star as a reservoir of heat energy $U$.
- Length of time to radiate this energy away:

$$\tau_{th} \approx U / L$$

- Virial theorem relates $U$ to star’s gravitational binding energy $\Omega$:

$$2U + \Omega = 0 \Rightarrow -\Omega = 2U$$

- We’ll see later that for a star:

$$\Omega = -q \frac{GM^2}{R}$$

- where $q$ is a dimensionless constant of order 1, so:

$$\tau_{th} = \frac{q}{2} \frac{GM^2}{LR} \approx 3 \times 10^7 \left( \frac{M}{M_{\text{Sun}}} \right)^2 \left( \frac{L_{\text{Sun}} R_{\text{Sun}}}{LR} \right) \text{ y.}$$
1.7c Nuclear timescale

- Radioisotope dating \( \Rightarrow \) Earth rocks much older than K-H “age” for Sun (oops...!)
- Energy available from thermonuclear fusion:
  \[
  E = mc^2 = 0.007q_{sc}XMC^2
  \]
  Mass fraction converted Schönberg-Chandrasekhar
- Nuclear timescale = time to radiate this amount of energy at main-sequence luminosity \( L \):
  \[
  t_{MS} = \frac{E}{L} = 0.007q_{sc}Xc^2 \frac{M}{L} \\
  \approx 7 \times 10^9 \frac{q_{sc}M}{(q_{sc}M)_{Sun}} \left( \frac{L}{L_{Sun}} \right) \text{y.}
  \]

Diffusion time

- If Sun were transparent, \( \gamma \)-rays generated in core would emerge in 2.3s with no energy loss.
- Actual escape time is much longer due to multiple absorptions & re-emissions.
- Total radiant energy content:
  \[
  U_r = u_rV = aT_c^4 \cdot \frac{4}{3} \pi R^3
  \]
  \[
  \text{Crude approximation: } \bar{T}_{Sun} \approx T_{c,Sun} \left/ 2 \right. = 7 \times 10^6 \text{ K.}
  \]
  \[
  \text{So time for a photon to diffuse out is roughly:}
  t_{diff} = \frac{U_r}{L} \approx 2 \times 10^5 \left( \frac{R}{R_{Sun}} \right)^3 \left( \frac{T_c}{T_{c,Sun}} \right)^4 \left( \frac{L_{Sun}}{L} \right) \text{y.}
  \]
2. Equations of stellar structure

- Object: find a set of $n$ equations in $n$ variables that describe the structure of a star.
- Augment them with $n$ boundary conditions.
- We’ll start with the basics like making sure that mass, momentum and energy are conserved.

2.1 Mass continuity

- Gotta conserve mass!
- By considering a shell of radius $r$ in a region of local density $\rho(r)$, we found in AS2001 that:

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

- This introduces 2 unknowns:
  - $M(r)$ = mass enclosed within sphere radius $r$
  - $\rho(r)$ = local gas density at radius $r$.
- So we need at least 1 more equation!
2.2 Hydrostatic equilibrium

- Otherwise known as force balance or conservation of momentum!
- Main forces operating are self-gravity and internal pressure.
- In AS 2001, considered force balance across a thin shell to get:
  \[ \frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \]
- So now we have a third unknown:
  - \( P(r) \) = local pressure at radius \( r \).
- But still only 2 equations so we need another!

\[ \frac{dL}{dm} = \varepsilon, \text{ and so } \frac{dL}{dr} = 4\pi r^2 \rho \varepsilon \]

2.3 Energy Conservation I: sources & sinks

- Energy moves through the star.
- Consider a shell of mass:
  \[ dm = \rho dV = 4\pi r^2 \rho dr \]
- Energy flows into bottom of shell at rate \( L(m) \)
- Energy leaves top of shell at rate \( L(m)+dL \)
- Denote rate at which energy is produced or absorbed within shell (per unit mass) by:
  \[ \frac{dL}{dm} = \varepsilon, \text{ and so } \frac{dL}{dr} = 4\pi r^2 \rho \varepsilon \]
- That’s one more equation but 2 extra unknowns:
  - \( L(r) \) = local luminosity at radius \( r \).
  - \( \varepsilon (r) \) = energy generation per unit mass at radius \( r \).
Energy sources and sinks

• We’ll consider these later. They include:
• Nuclear, e.g. \( 4p \rightarrow \alpha + 26.72 \text{ MeV} \)

• Neutrino, e.g. \( \gamma + e^- \rightarrow e^- + \gamma \)
  or \( \gamma + e^- \rightarrow e^- + \nu + \bar{\nu} \)

• Nonadiabatic expansion, i.e.
  \[
dQ = TdS = dE + PdV
\]

• Note that these imply a relationship between T, P and energy generation rate, i.e. another equation.