Stellar mass limits

- Hydrostatic equilibrium is the key element.
- Combine ideal classical gas law: \( P_c = \frac{\rho_c}{m} kT_c \).

- with expression for central pressure to get:
  \[
  kT_c = \left( \frac{\pi}{36} \right)^{1/3} GmM^{2/3} \rho_c^{1/3}.
  \]
- Applied to a contracting protostar, shows that \( T_c \) rises steadily as \( \rho_c \) increases.
- Temperature will continue to rise until either
  - Thermonuclear fusion starts to regulate \( T_c \); or
  - Electrons in core become degenerate
- Condition for stardom is that fusion sets in before electron degeneracy prevents further contraction.

Maximum core temperature

- Suppose star contracts until e⁻ are degenerate in core but ions form a classical gas. Then:
  \[
  P_c = K_{NR} n_e^{5/3} + n_i kT_c
  \]
  \[
  = K_{NR} \left( \frac{\rho}{m_H} \right)^{5/3} + \frac{\rho}{m_H} kT_c \text{ for a pure H plasma.}
  \]
- Use hydrostatic equilibrium to eliminate \( P_c \):
  \[
  kT_c = \left( \frac{\pi}{36} \right)^{1/3} Gm_H M^{2/3} \rho_c^{1/3} - K_{NR} \left( \frac{\rho_c}{m_H} \right)^{2/3}.
  \]
- Has form
  \[
  kT_c = A \rho_c^{1/3} - B \rho_c^{2/3} \text{ which has maximum value}
  \]
  \[
  (kT_c)_{\text{max}} = A^2 / 4B \text{ at } \rho_{c,\text{max}} = (A / 2B)^3.
  \]
Minimum mass of a star

- Substituting, get

\[(kT_c)_{\text{max}} = \left(\frac{\pi}{36}\right)^{2/3} \frac{G^2 m_H^{8/3}}{4K_{NR}} M^{4/3}.\]

- Need this max temp to be $> T_{\text{ign}}$, so:

\[M_{\text{min}} = \left(\frac{36}{\pi}\right)^{1/2} \left(\frac{4 K_{NR}}{G^2 m_H^{8/3}}\right)^{3/4} (kT_{\text{ign}})^{3/4}.\]

For $T_{\text{ign}} = 1.5 \times 10^6$ K, $M_{\text{min}} \approx 0.1M_{\text{Sun}}$.

Radiation pressure in stellar core

\[P_c = P_{\text{gas}} + P_{\text{rad}}, \] more conveniently written in terms of a parameter $\beta$:

\[P_{\text{gas}} = \beta P_c = \frac{P_c}{m} k T_c \quad \text{and} \quad P_{\text{rad}} = (1 - \beta) P_c = \frac{1}{3} a T_c^4.\]

Use $\beta$ to eliminate $T_c$:

\[P_c = \left[\frac{3 (1 - \beta)}{\alpha} \frac{k \rho_c}{m \beta^4}\right]^{1/3}.\]

Equate this to hydrostatic equilibrium pressure:

\[\left[\frac{\pi}{36}\right]^{1/3} G M^{2/3} = \left[\frac{3 (1 - \beta)}{\alpha} \frac{k \rho_c}{m \beta^4}\right]^{1/3}.\]
Maximum mass of a star

- Can use this equation to plot radiation pressure fraction \((1-\beta)\) as function of stellar mass.
- When \((1-\beta) > 0.5\) or so, radiation pressure destabilizes star -- so stars with \(M > 50 \, M_{\text{Sun}}\) or so tend to blow apart

Understanding the main sequence

- Develop dimensionless equations of stellar structure for homologous models.
- Discover how mass-radius and mass-luminosity relations depend on opacity and energy generation.
- Determine slope of main sequence.
- Learn how changes in abundance affect location of main sequence in HR diagram.
Dimensionless structure equations

• Define \( x = \frac{r}{R}, \quad q = \frac{M(r)}{M}, \quad t = \frac{T}{T_0}, \quad p = \frac{P}{P_0}. \)

• and use \( \rho = \frac{\mu m P}{k T}. \)

• Hydrostatic equilibrium becomes:

\[
\frac{P_0 dp}{R dx} = \frac{\mu m P_0 P GM q}{k T_0 t R^2 x^2} \Rightarrow \frac{dp}{dx} = \frac{p q}{t x^2}
\]

where \( T_0 = \frac{\mu m GM}{k R}. \)

Mass continuity + energy transport

• Similarly:

\[
\frac{M dq}{R dx} = 4\pi R^2 x^2 \frac{\mu m P_0 P}{k T_0 t} \Rightarrow \frac{dq}{dx} = x^2 \frac{P}{t}
\]

where \( P_0 = \frac{GM^2}{4\pi R^4}. \)

• Transport: Let \( \kappa = \kappa_0 \rho^\alpha T^{-\beta} = \kappa_0 \left( \frac{\mu m P_0 P}{k T_0 t} \right)^\alpha (T_0 t)^{-\beta}. \)

\[
\frac{T_0 dt}{R dx} = -\frac{3\kappa_0}{16\pi ac(T_0 t)} \left( \frac{\mu m P_0 P}{k T_0 t} \right)^{1+\alpha} (T_0 t)^{-\beta} \frac{L f}{R^2 x^2}
\]

\[
\Rightarrow \frac{dt}{dx} = -C \frac{P^{1+\alpha} f}{x^2} \quad \text{where} \quad C = \frac{3\kappa_0}{16\pi ac} \left( \frac{\mu m}{k} \right)^{1+\alpha} \frac{P_0^{1+\alpha} L}{T_0^{5+\alpha+\beta} R}. \]
Energy generation

- Energy generation:

  Let \( \varepsilon = \varepsilon_0 \rho T^\nu = \varepsilon_0 \frac{\mu m P_0}{k T_0} (T_0 t)^\nu \).

  \[
  \frac{L}{R} \frac{df}{dx} = 4\pi^2 \varepsilon \rho = 4\pi R^2 x^2 \varepsilon_0 \left( \frac{\mu m P_0}{k T_0} \right)^2 (T_0 t)^\nu
  \]

  \[
  \Rightarrow \frac{df}{dx} = D \frac{x^2 P^2}{t^{2-\nu}} \quad \text{where } D = \frac{4\pi \varepsilon_0 R^3}{L} \left( \frac{\mu m}{k} \right)^2 \frac{P_0^2}{T_0^{2-\nu}}.
  \]

Mass-radius relation: 1

- The constants \( P_0, T_0, C \) and \( D \) are unique to each star and apply throughout the entire structure.

- The product \( CD \) eliminates \( L \) to give a direct relation between mass and radius:

  \[
  CD = \frac{3\kappa \varepsilon_0 R^2}{4 ac} \left( \frac{\mu m}{k} \right)^{3+\alpha} \frac{P_0^{3+\alpha}}{T_0^{7+\alpha+\beta-\nu}}
  \]

  \[
  = \frac{3\kappa \varepsilon_0 R^2}{4 ac} \left( \frac{\mu m}{k} \right)^{3+\alpha} \left( \frac{GM^2}{4\pi R^4} \right)^{3+\alpha} \left( \frac{\mu m GM}{R} \right)^{\nu-7-\alpha-\beta}
  \]

  \[
  = \frac{3\kappa \varepsilon_0}{4 ac} \left( \frac{m}{k} \right)^{3+\alpha} \left( \frac{G}{4\pi} \right)^{3+\alpha} \left( \frac{mG}{k} \right)^{\nu-7-\alpha-\beta} \mu^{\nu-4-\beta} \frac{M^{\nu+1+\alpha-\beta}}{R^{\nu+3+3\alpha-\beta}}
  \]
Mass-radius relation: 2

- Hence get mass-radius relation:
  \[ R^{\nu+3+3\alpha-\beta} \propto M^{\nu-1+3\alpha-\beta}. \]

- Kramers’ opacity law and CNO cycle give \( \alpha=1, \beta=7/2, \nu=16 \), so:
  \[ R \propto M^{12.5/18.5}. \]

- Böhm-Vitense finds that in practice, an opacity law with \( \alpha=0.5, \beta=2.5 \) gives a better fit:
  \[ R \propto M^{13/18}. \]