2.4 Energy conservation II: Transport.

- The Sun’s interior is hotter than its surface.
- Existence of a temperature gradient implies an outward flux of energy.
- Energy flux is determined by conservation of energy as just shown.
- Temperature gradient depends on method of energy transport:
  - Radiative diffusion
  - Convective motions

Flux and radiation pressure

- From AS 3002, flux through surface element dA in frequency interval from $\nu$ to $\nu + d\nu$ is:

$$F_{\nu} = \int I_{\nu} \cos \theta \, d\Omega = \frac{2\pi}{4\pi} \int_{-1}^{1} I_{\nu} \mu \, d\mu$$

where we have substituted $\mu = \cos \theta$.

- Each photon carries momentum $p_{\nu} = E/c = h\nu/c$
- Bounces off dA at angle of incidence $\cos \theta$.
- Momentum transferred per photon= $2 \, p_{\nu} \cos \theta$. 
Pressure = momentum flux

- Pressure = outward (photons/sec/unit area) 
  \[ x (2 \rho_v \cos \theta) \]
  = (in+out)(photons/sec/unit area) 
  \[ x ( \rho_v \cos \theta) \]
  \[ P_{\text{rad}, \nu} = \frac{1}{c} \int I_v \cos \theta dΩ = \frac{2\pi}{c} \int I_v \mu^2 d\mu \]

- The factor \( \cos^2 \theta \) allows for both foreshortening of the surface element’s cross-section and transfer of normal component of momentum.

Radiative energy transport

- Specific intensity of beam travelling at angle \( \theta \) to radial direction in medium of density \( \rho \), opacity \( \kappa_v \) and source function \( S_v \):
  \[ \cos \theta \frac{dI_v}{dr} = \rho \kappa_v (S_v - I_v) \]. Multiply both sides by \( \mu = \cos \theta \), and integrate:
  \[ \int_{-1}^{1} \mu^2 \frac{dI_v}{dr} d\mu = \rho \kappa_v \int_{-1}^{1} \mu d\mu (S_v - I_v) \]
  \[ \Rightarrow \frac{d}{dr} \int_{-1}^{1} I_v \mu^2 d\mu = \rho \kappa_v S_v \int_{-1}^{1} \mu d\mu - \rho \kappa_v \int_{-1}^{1} I_v \mu d\mu \]
  cf. Radiation pressure
  cf. Flux
Radiative-equilibrium temperature gradient

- We find that the opacity determines the temperature gradient:

\[ \frac{dP_{\text{rad},\nu}}{dr} = -\kappa_{\nu}\rho F_{\nu} \Rightarrow \frac{dP_{\text{rad}}}{dr} = -\frac{\kappa P}{c} F \]

where we define a flux-weighted mean opacity,

\[ \kappa \equiv \int_{0}^{\infty} \kappa_{\nu} F_{\nu} d\nu / \int_{0}^{\infty} F_{\nu} d\nu. \] But \( P_{\text{rad}} = \frac{u}{3} = \frac{1}{3} aT^{4}, \)

so \( \frac{dP_{\text{rad}}}{dr} = \frac{4}{3} aT^{3} \frac{dT}{dr}. \) Also \( F = \frac{L}{4\pi r^{2}}, \)

\[ \Rightarrow \frac{dT(r)}{dr} = -\frac{3\kappa P}{4acT^{3}} \frac{L(r)}{4\pi r^{2}} \]

Another equation and a new variable, \( T(r). \)

Convective equilibrium

- Suppose temperature gradient is radiative.
- Is it stable to small local perturbations?
- Suppose a blob of mass \( \delta m \) at radius \( r \) has its temperature perturbed by a small amount:

\[ \Delta T(r) = T_{\delta m}(r) - T(r). \]

- Pressure will change by

\[ \Delta P(r) = P_{\delta m}(r) - P(r). \]
- but pressure balance is quickly restored by a change in volume, to give density difference from surroundings:

\[ \Delta \rho(r) = \rho_{\delta m}(r) - \rho(r). \]
- Temperature excess with pressure equilibrium in ideal gas \( \rightarrow \) density deficiency \( \rightarrow \) buoyancy.
Buoyant stability

• If $\Delta T > 0$, buoyant force > gravity, so blob rises to new position at $r + \Delta r$.

• Surroundings at new position have density

$$\rho(r) + \frac{d\rho}{dr} \Delta r$$

• while blob changes density to match local pressure:

$$\rho_{\delta m}(r) + \left(\frac{d\rho}{dr}\right)_{\delta m} \Delta r$$

• Element is stable if it becomes denser than surroundings, i.e. if:

$$\left| \frac{d\rho}{dr} \right| > \left| \frac{d\rho}{dr} \right|_{\delta m}$$

• Ideal gas:

$$P \propto \rho T \Rightarrow \left| \frac{dT}{dr} \right| < \left| \frac{dT}{dr} \right|_{\delta m}$$

for stability.

Adiabatic changes

• Rising blob is hotter than its surroundings as it rises so can only lose heat (& vice versa for falling blob).

• Hence change in temperature with $r$ must be less than adiabatic (no heat loss) value:

$$\left| \frac{dT}{dr} \right|_{\delta m} < \left| \frac{dT}{dr} \right|_{ad}$$

• Adiabatic gradient is given by:

$$PV^\gamma = \text{const} \Rightarrow P \propto \rho^\gamma \propto \left(\frac{P}{T}\right)^\gamma$$

where $\gamma = \frac{C_P}{C_V}$

$$\Rightarrow P^{1-\gamma}T^\gamma = \text{const}$$

$$\Rightarrow (1 - \gamma) \frac{dP}{dr} + \gamma \frac{P dT}{T dr} = 0.$$
Logarithmic T-P gradients

- For an adiabatic blob, we thus get:
  \[ \frac{1}{T} \left( \frac{dT}{dr} \right)_{ad} = \frac{\gamma - 1}{\gamma} \frac{1}{P} \left( \frac{dP}{dr} \right) \]

- or:
  \[ \left( \frac{d \log T}{d \log P} \right)_{ad} = \frac{\gamma - 1}{\gamma} \]

- Use hydrostatic equilibrium to get a similar T-P relation for the radiative gradient:
  \[ \left( \frac{d \log T}{d \log P} \right)_{rad} = \frac{3kL(r)P}{16\pi acT^4GM} \]

Convective stability criterion

- Remembering that pressure is the same inside and outside blob at all times, we can write the stability criterion as:

  \[ \left| \left( \frac{d \log T}{d \log P} \right)_{rad} \right| < \left| \left( \frac{d \log T}{d \log P} \right)_{ad} \right| \]

  \[ \Rightarrow \frac{3kL(r)P}{16\pi acT^4GM} < \frac{\gamma - 1}{\gamma} \]
Convectively unstable regions

• (1) Cores of massive stars:
  Radiation flux $L(r)/4\pi r^2$ can become very large while opacity $\kappa\rho$ remains small in the centres of main massive main-sequence stars.

• (2) Outer envelopes of cool stars:
  Adiabatic exponent $\gamma$ can approach unity in sub-surface ionization zones in cool stars.
  Hence $(\gamma - 1)/\gamma$ can become small, and convection will set in at quite low values of $|dT/dr|_{\text{rad}}$.

Energy transport

• In formulating stellar structure problem, use a single expression for the temperature gradient:

$$\frac{d\log T}{d\log P} = (1 - \xi)\left(\frac{d\log T}{d\log P}\right)_{\text{rad}} + \xi\left(\frac{d\log T}{d\log P}\right)_{\text{ad}}.$$

where $\xi$ characterizes the convective efficiency:

$\xi = 0 \Rightarrow$ radiative equilibrium

$\xi = 1 \Rightarrow$ adiabatic convection

$0 < \xi < 1 \Rightarrow$ non-adiabatic convection: $\xi$ must be determined from convection theory.
2.5.1 Constitutive relations

- Still need additional equations to describe $\rho$, $\varepsilon$, $\kappa$, $\xi$ and $(\log T/\log P)$ in terms of:
  - the state variables $T$ and $P$, and
  - the composition of the stellar material ($X, Y, Z$ or $X_i$)

- The following constitutive relations close the system of ODEs:
  \[
  \begin{align*}
  \rho &= \rho(P, T, X_i) \quad \text{(equation of state)} \\
  \varepsilon &= \varepsilon(\rho, T, X_i) \quad \text{(nuclear energy generation rate)} \\
  \kappa &= \kappa(\rho, T, X_i) \quad \text{(opacity)} \\
  \xi &= \xi(\rho, T, X_i) \quad \text{(convective efficiency)} \\
  \frac{d \log T}{d \log P}(\rho, T, \xi, \kappa, X_i) &= \text{(energy transport)}
  \end{align*}
  \]

2.5.2 Equations of stellar structure

- We have now determined the four basic (time-independent) equations of stellar structure.
- Use mass continuity to transform them to have enclosed mass as the independent variable.

- Mass continuity:
  \[
  \frac{dr}{dm} = \frac{1}{4\pi^2 \rho}
  \]

- Conservation of energy:
  \[
  \frac{dL}{dm} = \varepsilon
  \]

- Hydrostatic equilibrium:
  \[
  \frac{dP}{dm} = -\frac{Gm}{4\pi r^4}
  \]

- Energy transport:
  \[
  \frac{dT}{dm} = -\frac{Gm T}{4\pi r^4} \frac{d \log T}{d \log P}
  \]
2.5.3 Boundary conditions

- To solve a system of n ODEs, we need to specify n boundary conditions.
- In Lagrangian frame, boundaries are at the centre \((m=0)\) and the surface \((m=M)\).

At the centre:

\[
r(m = 0) = 0 \\
L(m = 0) = 0
\]

At the surface:

\[
T(m = M) = T_{\text{eff}} \left( \frac{L}{4\pi R^2 \sigma} \right)^{1/4} \\
P_{\text{gas}}(m = M) = 0
\]

2.5.4 Solution

- Solution of equations of stellar structure gives the run of \(P, T, m\) and \(L\) as functions of \(r\) throughout the domain \(0 < r < R\).
- Solutions are characterized uniquely by
  - Total mass of star \(M = m(R)\)
  - Run of chemical composition through star.
  - Gravitational binding energy.
- Gives quantitative description of stellar interior.