3. Stellar models I

- **Numerical solutions:**
  - **Shooting method**
  - **Difference method**
- **Analytical approximations:**
  - **Homologous models**

3.1 Shooting method

- **Divide star into two zones, inner and outer.**
- **Estimate additional boundary conditions**
- **Inward solution:**
  - **At $m=M$ we have $P_s=0$, $T_s=T_{eff}$**
  - **Estimate R and L**
  - **Integrate inwards to fitting point m_f**
  - **Hence obtain P_{if}, T_{if}, L_{if}, r_{if}**
- **Outward solution:**
  - **At $m=0$ we have $R_c=0$, $L_c=0$**
  - **Estimate P_c and T_c**
  - **Integrate outwards to fitting point m_f**
  - **Hence obtain P_{of}, T_{of}, L_{of}, r_{of}**
- **Adjust till conditions match at fitting point.**

Matching at fitting point

- **In general:**

$$\delta P_f = P_{if} - P_{of} \neq 0,\ \ \delta T_f = T_{if} - T_{of} \neq 0,$$
$$\delta L_f = L_{if} - L_{of} \neq 0,\ \ \delta r_f = r_{if} - r_{of} \neq 0.$$

- **Repeat inward solution:**
  - **with R+δR, L**
  - **with R, L+δL**
- **Repeat outward solution:**
  - **with $P_c+\delta P_c, T_c$**
  - **with $P_c, T_c + \delta T_c$**
- **From resulting changes in r_{if}, r_{of} etc form derivatives:**

$$\frac{\partial \delta r_f}{\partial R},\ \frac{\partial \delta r_f}{\partial L},\ \frac{\partial \delta r_f}{\partial P_c},\ \text{etc.}$$

Differential corrections

- **Now correct the original estimates for R, L, P_c, T_c by solving:**

$$\begin{pmatrix} \frac{\partial \delta r_f}{\partial R} & \frac{\partial \delta r_f}{\partial L} & \frac{\partial \delta r_f}{\partial P_c} & \frac{\partial \delta r_f}{\partial T_c} \\ \frac{\partial \delta L_f}{\partial R} & \frac{\partial \delta L_f}{\partial L} & \frac{\partial \delta L_f}{\partial P_c} & \frac{\partial \delta L_f}{\partial T_c} \\ \frac{\partial \delta P_f}{\partial R} & \frac{\partial \delta P_f}{\partial L} & \frac{\partial \delta P_f}{\partial P_c} & \frac{\partial \delta P_f}{\partial T_c} \\ \frac{\partial \delta T_f}{\partial R} & \frac{\partial \delta T_f}{\partial L} & \frac{\partial \delta T_f}{\partial P_c} & \frac{\partial \delta T_f}{\partial T_c} \end{pmatrix} \begin{pmatrix} \delta R \\ \delta L \\ \delta P_c \\ \delta T_c \end{pmatrix} = \begin{pmatrix} \delta r_f \\ \delta L_f \\ \delta P_f \\ \delta T_f \end{pmatrix}$$

- **for δR, δL, δP_c, δT_c . Repeat until $\delta P_f < \varepsilon$, etc where ε is some suitably small number.**

3.2 Difference method

- **Alternative numerical approach:**
- **Write ODEs for stellar structure as difference equations:**

$$\frac{r_{j+1}-r_j}{m_{j+1}-m_j}=\left(\frac{1}{4\pi r^4\rho}\right)_{j+1/2}.$$

- **Subscript j refers to variable values at position j inside star.**
- **Divide star into J+1 shells from m_0 = 0 to m_J = M:**
  - **4 nonlinear equations for J shells**
  - **4 boundary equations**
  - **4J unknowns**
- **Linearize resulting 4J nonlinear simultaneous eqs in 4J unknowns and solve (using, e.g. Newton-Raphson).**

3.3 Homologous stars

- **How do properties of stars vary with mass?**
- **Use simplifying assumptions to avoid detailed numerical treatment of full problem.**
- **Look at family of models:**
  - **in complete equilibrium**
  - **each related to others by a change of scale**
- **Define a family of *homologous* models, i.e. having same radius, mass distributions relative to reference model denoted by subscript 0:**

$$r=\frac{R}{R_0}r_0 \quad \Rightarrow \frac{dr}{dr_0}=\frac{R}{R_0} \qquad \text{and}$$

$$m=\frac{M}{M_0}m_0 \Rightarrow \frac{dm}{dm_0}=\frac{M}{M_0}$$

3.3.1 Mass continuity

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \Rightarrow \rho \propto \frac{1}{r^2 (dr/dm)}$$

$$\frac{\rho}{\rho_0} \propto \frac{(dr_0/dm_0)}{(dr/dm)} \left(\frac{r}{r_0}\right)^{-2}$$

$$\propto \frac{(dm/dm_0)}{(dr/dr_0)} \left(\frac{r}{r_0}\right)^{-2}$$

$$\propto \frac{(M/M_0)}{(R/R_0)^3} \Rightarrow \rho \propto \frac{M}{R^3}$$

3.3.2 Hydrostatic equilibrium

- **Can use H.E. similarly to show that:**

$$P \propto \frac{M^2}{R^4}$$

- **Write equation of state as a power law:**

$P = P_0 \rho^{\chi_\rho} T^{\chi_T}$ where the constants P_0, χ_ρ, χ_T are assumed to be the same for all stars in the homologous family.

- **Equate the two expressions for P above and differentiate:**

$$4d\ln R + \chi_\rho d\ln \rho + \chi_T d\ln T = 2d\ln M.$$

3.3.3 Energy and transport equations

- **Treat in same way as mass continuity and hydrostatic equilibrium equations**
- **Adopt power laws for dependence of energy generation and opacity on local density and temperature:**

$$\varepsilon = \varepsilon_0 \rho^{\lambda} T^{\nu}$$

$$\kappa = \kappa_0 \rho^{n} T^{-s}$$

3.3.4 Power-law relations

- **Can we predict how stellar properties vary with mass using an analytic treatment?**
- **i.e. we want to construct relations of the form:**

$$R \propto M^{\alpha_R} \Rightarrow d\ln R = \alpha_R d\ln M$$

$$\rho \propto M^{\alpha_\rho} \Rightarrow d\ln \rho = \alpha_\rho d\ln M$$

$$T \propto M^{\alpha_T} \Rightarrow d\ln T = \alpha_T d\ln M$$

$$L \propto M^{\alpha_L} \Rightarrow d\ln L = \alpha_L d\ln M$$

- **Substitute these into expression found in Section 3.3.2 for hydrostatic equilibrium:**

$$4d\ln R + \chi_\rho d\ln \rho + \chi_T d\ln T = 2d\ln M$$

$$\Rightarrow 4\alpha_R + \chi_\rho \alpha_\rho + \chi_T \alpha_T = 2$$