5. Equation of state

- Stellar material is an almost perfect gas.
- Main differences from laboratory gas:
  - Ionized: plasma allows greater compression ($10^{-15}$ m cf. $10^{-10}$ m).
  - In TE with radiation $\Rightarrow$ intensity follows Planck’s law.
  - Particles may not be classical: quantum mechanical effects.
  - Particles may be relativistic: special relativity.
- A complete description of the macroscopic properties of the gas requires 3 state variables.
- First Law of Thermodynamics:
  \[ dE = TdS - PdV + \mu dN \]
  Change in internal energy

Spin and polarization

- Need to allow for intrinsic angular momentum or spin of particles, or for different polarizations of photons:
  \[ g(p)dp = g_s \frac{V}{h^3} 4\pi p^2 dp \]
  Partition function

<table>
<thead>
<tr>
<th>Particle type</th>
<th>Spin</th>
<th>$g_s$</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>p, n, e</td>
<td>1/2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>1/2</td>
<td>1</td>
<td>Only 1 polarization</td>
</tr>
<tr>
<td>photons</td>
<td>1</td>
<td>2</td>
<td>2 independent polarizations for EM wave.</td>
</tr>
</tbody>
</table>
5.2 Occupation probabilities and Internal energy

- In TE, macroscopic state variables $T$, $P$, $\mu$ determine equilibrium distribution of particles in quantum states. Different for fermions, bosons.
- Energy of particle, mass $m$, in quantum state with momentum $\mathbf{p}$, is:
  \[ \varepsilon_p^2 = \mathbf{p}^2 c^2 + m^2 c^4 \]
- Occupation probability $f(\varepsilon_p)$ is the average number of particles in a state with energy $\varepsilon_p$:
  \[ \Rightarrow \text{Internal energy } E = \int_0^\infty \varepsilon_p f(\varepsilon_p) g(p) dp. \]
  Total number of particles in gas $N = \int_0^\infty f(\varepsilon_p) g(p) dp$.

5.3 Pressure in an ideal gas – 1

\[ P = -\frac{\partial E}{\partial N} \bigg|_{N,S} = -\int_0^\infty \frac{\partial \varepsilon_p}{\partial N} f(\varepsilon_P)g(p) dp. \]

Use \[ \frac{\partial \varepsilon_p}{\partial N} = \frac{\partial \varepsilon_p}{\partial p} \frac{\partial p}{\partial N}. \] Since $V = L^3$ and $p \propto L$, get

\[ p \propto V^{-1/3} \Rightarrow \frac{dp}{dV} = -\frac{p}{3V}. \]

Also $\varepsilon_p^2 = p^2 c^2 + m^2 c^4$ \[ \Rightarrow 2\varepsilon_p \frac{\partial \varepsilon_p}{\partial p} = 2pc^2 \]

\[ \Rightarrow \frac{\partial \varepsilon_p}{\partial p} = \frac{pc^2}{\varepsilon_p} \equiv v_p, \text{ so: } \frac{\partial \varepsilon_p}{\partial N} = -\frac{pv_p}{3V}. \]

Speed of particle with momentum $p$. 

Page 2
Pressure in an ideal gas – 2

Hence \( P = \frac{1}{3V} \int_{0}^{\infty} pv_p f(\varepsilon_p) g(p) dp = \frac{N}{3V} \langle pv_p \rangle \). \text{Average over N particles in gas}

Non-relativistic: \( \varepsilon_p = mc^2 + \frac{p^2}{2m} \), and \( v_p = \frac{p}{m} \)

\[ \Rightarrow P = \frac{2N}{3V} \left\langle \frac{p^2}{2m} \right\rangle = \frac{2}{3} \text{ of kinetic energy density.} \]

Ultra-relativistic: \( \varepsilon_p = pc \), and \( v_p = c \)

\[ \Rightarrow P = \frac{N}{3V} \langle pc \rangle = \frac{1}{3} \text{ of kinetic energy density.} \]

5.4 Equation of state for an ideal classical gas – 1

\[ P = \frac{1}{3V} \int_{0}^{\infty} pv_p f(\varepsilon_p) g(p) dp \]

Substitute for density of states:

\[ g(p) dp = g_s \frac{V}{h^3} 4\pi p^2 dp \]

Occupation probability in classical limit:

\[ f(\varepsilon_p) \approx \frac{1}{\exp[(\varepsilon_p - \mu)/kT]} \ll 1 \]
Equation of state for an ideal classical gas – 2

\[ P = \frac{1}{3V} \exp \left( \frac{\mu}{kT} \right) \int_0^\infty p v_p \exp \left( \frac{-\epsilon_p}{kT} \right) g_s \frac{V}{h^3} 4\pi p^2 dp. \]

Now \( d\epsilon_p = v_p dp \), so can rewrite integral:

\[ \int_0^\infty p^3 \exp \left( \frac{-\epsilon_p}{kT} \right) v_p dp = -kT \int_0^\infty p^3 \left( \exp \left( \frac{-\epsilon_p}{kT} \right) \right) dp. \]

Or by parts:

\[ = 3kT \int_0^\infty \exp \left( \frac{-\epsilon_p}{kT} \right) p^2 dp. \]

Equation of state for an ideal classical gas – 3

• Substitute back to get expression for the pressure in a classical ideal gas:

\[ P = \frac{kT}{V} \exp \left( \frac{\mu}{kT} \right) \int_0^\infty \exp \left( \frac{-\epsilon_p}{kT} \right) g_s \frac{V}{h^3} 4\pi p^2 dp. \]

• Comparing this with expression for the total number of particles:

\[ N = \exp \left( \frac{\mu}{kT} \right) \int_0^\infty \exp \left( \frac{-\epsilon_p}{kT} \right) g_s \frac{V}{h^3} 4\pi p^2 dp \]

• leads to the equation of state for an ideal classical gas:

\[ P = \frac{N}{V} kT = nkT. \]
Condition for ideal classical gas

- In classical limit:
  \[ \exp[(mc^2 - \mu)/kT] >> 1. \]
- Use this to derive an explicit expression for the chemical potential of a classical gas. In equation for total no. of particles substitute:
  \[ \varepsilon_p = mc^2 + \frac{p^2}{m} \]
  and integrate to get:
  \[ N = \exp\left(\frac{\mu - mc^2}{kT}\right) g_s \frac{V}{h^3} (2\pi mkT)^{3/2}. \]
- Rearrange:
  \[ \mu - mc^2 = -kT \ln\left(\frac{g_s n_Q}{n}\right) \]
  \[ n_Q = \left[\frac{2\pi mkT}{\hbar^2}\right]^{3/2} \]
  Quantum concentration
  \[ n = \frac{N}{V} \]

Ultra-relativistic classical gas

- For UR particles neglect rest energy \( mc^2 \):
  Substitute \( \varepsilon_p = pc \) into expression for \( N \) to get:
  \[ \mu = -kT \ln\left(\frac{g_s n_Q}{n}\right) \text{ where } n_Q = 8\pi \left[\frac{kT}{\hbar c}\right]^3. \]
- Hence condition for classical gas
  \[ \exp[(mc^2 - \mu)/kT] >> 1 \]
  is satisfied if \( n << n_Q \).
Fermi-Dirac distribution

- Total number of particles:
  \[ N = \int g_s \frac{V}{\hbar^3} 4\pi p^2 dp = \frac{8\pi V}{3\hbar^3} p_F^3 \] where \( g_s = 2 \).

- Rearrange to get Fermi momentum:
  \[ p_F = \hbar \left[ \frac{3n}{8\pi} \right]^{1/3} \] where \( n = \frac{N}{V} \) as usual.

- To get equation of state evaluate internal energy.
  Non-relativistic case: \( p_F << mc \) implies \( n << (mc/\hbar)^3 \) (\( = (1/\lambda_\nu)^3 \)). Subst into expression for \( E \) to get:
  \[ E = \int_0^{p_F} \epsilon_p g_s \frac{V}{\hbar^3} 4\pi p^2 dp = N \left[ mc^2 + \frac{3p_F^2}{10m} \right]. \]

Equation of state

- NR degenerate electron gas: neglect \( mc^2 \) and recall that \( P = (2/3)E \):
  \[ P = K_{NR} n^{5/3}, \text{ where } K_{NR} = \frac{h^3}{5m} \left[ \frac{3}{8\pi} \right]^{2/3} \]

- UR degenerate electron gas:
  \[ P = K_{UR} n^{4/3}, \text{ where } K_{UR} = \frac{hc}{4} \left[ \frac{3}{8\pi} \right]^{1/3}. \]

- Note that in both cases, pressure depends only on density, not on temperature.
5.6 Photon gas

- Thermal radiation may be described as gas of zero-mass bosons with zero chemical potential.
- Photon number density:
  \[ n = b T^3 \]
  where \( b = 2.404 \times 8\pi (k / hc)^3 = 2.03 \times 10^7 \text{ K}^{-3} \text{ m}^{-3} \).
- Internal energy density:
  \[ U = a T^4 \]
  where \( a = 2.404 \times 8\pi^5 k^4 / 15 (hc)^3 = 7.565 \times 10^{-16} \text{ J K}^{-4} \text{ m}^{-3} \).
- Radiation pressure:
  \[ P_{\text{rad}} = U / 3 = a T^4 / 3. \]

Combining pressures

- Total pressure from all components of a plasma consisting of a gas+radiation mix:
  \[ P_t = P_{\text{gas}} + P_{\text{rad}} = P_{\text{ion}} + P_{\text{e}} + P_{\text{rad}} \]
- Recall that internal temp of star \( T_1 \sim M/R \) and particle density \( n \sim M/R^3 \).
- Pressure ratio (classical gas/radiation):
  \[ \frac{P_{\text{rad}}}{P_{\text{gas}}} = \frac{a T_1^4 / 3}{n_e k T_1 + n_i k T_1} \propto \frac{M^4 / R^4}{M^2 / R^4} \propto M^2. \]
- Hence radiation pressure becomes increasingly important for increasing stellar mass, leading ultimately to instability at \( M \sim 50 \text{ M}_\odot \).
5.7 Density-temperature diagram

Classical, UR
\[ P = nkT \]

Classical, NR
\[ P = n k T \]

Degenerate, NR

\[ P = K_{NR} n^{5/3} \]

Degenerate, UR

\[ P = K_{UR} n^{4/3} \]

Core of supernova progenitor

Sun

White dwarf

Normal metal

Electron concentration \( n \) (m\(^{-3}\))

Temperature \( T \) (K)

Core of supernova progenitor

Sun

White dwarf

Normal metal

Electron concentration \( n \) (m\(^{-3}\))

Temperature \( T \) (K)