SH ASTRONOMY AND ASTROPHYSICS: AS 3003

Stellar Physics 2: Stellar interiors

1. During the first phases of protostellar collapse, the gravitational energy released is absorbed first by dissociation of H_2 (dissociation energy per molecule $\epsilon_D = 4.5$ eV), and then by ionization of atomic H (ionization energy per atom $\epsilon_I = 13.6$ eV).

   (i) State the timescale on which this stage of collapse will occur, and the timescale on which subsequent evolution will then proceed.

   (ii) Use the Virial theorem to show that the mean kinetic energy of the protons and electrons in the fully-ionized star immediately following the initial collapse is given by,

   $$kT = \frac{GMm_H}{6R}.$$

   (You may assume the star is composed of pure hydrogen.)

   (iii) Show further that, provided that a large decrease in radius has taken place during the collapse, the average internal temperature of the star will be of the order,

   $$kT = \frac{\epsilon_D + 2\epsilon_I}{12},$$

   once Virial equilibrium is re-established following ionization. Hence evaluate the average internal temperature T in Kelvin and the radius R in metres of a protostar with a mass of 1 $M_\odot$.

2. Given that the radiative pressure gradient is related to the radiative flux F in a stellar layer of density ρ and opacity κ at radius r by,

   $$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\kappa\rho}{c}F,$$

   show that the temperature gradient in radiative equilibrium is given by,

   $$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\kappa\rho}{4acT^3}\frac{L(r)}{4\pi r^2}.$$

   Use the equation of hydrostatic equilibrium to show that the logarithmic temperature and pressure gradients are related by

   $$\frac{\mathrm{d}\log T}{\mathrm{d}\log P} = \frac{3\kappa L(r)P}{16\pi acT^4GM}.$$

   A thermally isolated blob of material undergoes a small temperature increase relative to its surroundings and begins to rise under the influence of buoyancy. Show that the temperature inside the blob will vary as,

   $$(1-\gamma)\frac{\mathrm{d}P}{\mathrm{d}r} + \gamma\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}r} = 0,$$

   where γ is the ratio of specific heats. Hence derive a criteria for the layer to be stable against convection.

3. The temperature gradient in a star is given by the usual expression,

$$\frac{\partial T}{\partial r} = -\frac{3\kappa\rho}{4acT^3}\frac{L(r)}{4\pi r^2}.$$

(i) Transform this equation into Lagrangian form (i.e. with enclosed mass m as the independent variable).

(ii) An expression for the rate of change of luminosity with mass is,

$$\frac{\partial L}{\partial m} = \epsilon - c_V\frac{\partial T}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}.$$

(note: this is more general that the form given in the lectures). Hence, show that the temperature in the star obeys a diffusion equation,

$$\frac{\partial}{\partial m}\left(\sigma^*\frac{\partial T}{\partial m}\right) = c_V\frac{\partial T}{\partial t} - \left[\epsilon + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\right],$$

and find an expression for σ^*.

(iii) A diffusion equation of the form,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial T}{\partial x}\right),$$

has a *diffusion time* $\tau = X^2/D$, where X is a characteristic scale. Hence, write down an expression for the *thermal adjustment time* of a star that is out of thermal equilibrium. Note : this is only subtly different from the Kelvin-Helmholtz timescale.

4. Neutron stars are compact enough that the effects of general relativity become important. The pressure and density for a spherically symmetric, non-rotating star, are then given by 3 ordinary differential equations,

$$\frac{\kappa P}{c^2} = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \quad (1)$$

$$\frac{\kappa P}{c^2} = \frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{1}{2}\nu'^2 + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) \quad (2)$$

$$\kappa\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \quad (3)$$

where ν and λ are functions of r, the primes denote derivatives with respect to r, and $\kappa = 8\pi G/c^2$ (**not** the opacity!).

(i) Show that the third equation can be written in the form,

$$\kappa m = 4\pi r(1 - e^{-\lambda})$$

where,

$$m = \int_0^r 4\pi r^2 \rho \mathrm{d}r.$$

(ii) By differentiating the first of these equations with respect to r, and then eliminating the functions λ, λ', ν' and ν'', derive the relativistic generalization of the hydrostatic equilibrium equation,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm}{r^2}\rho\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}.$$

This is the *Tolman-Oppenheimer-Volkoff* equation.

(iii) In the Newtonian limit $c \to \infty$, and we easily recover the usual expression. For weak fields, expand the product of the brackets and retain only terms linear in $1/c^2$. Hence, derive the *post-Newtonian* approximation to the relativistic hydrostatic equilibrium equation.