Monte Carlo Radiation Transfer

- Introduce what’s needed to make an isotropic scattering code, uniform density medium
- Monte Carlo “Photon packets” and interactions
- Sampling from probability distributions
- Optical depths, isotropic emission & scattering
Monte Carlo Radiation Transport Basics

- Emit luminosity packet
- Packet travels some distance
- Something happens…

- Scattering, absorption, re-emission
Luminosity Packets

Total luminosity = $L$ (Watts, J/s, erg/s)

Each packet carries energy $E_i = L \Delta t / N$

$N =$ number of Monte Carlo packets

$\Delta t$ is time interval over which simulation being performed

MC packet represents $N_\gamma$ real photons, where $N_\gamma = E_i / h\nu_i$
Specific Intensity

\[ dE_\nu = I_\nu \cos \theta \, dA \, dt \, d\nu \, d\Omega \]

Units of \( I_\nu \): J/m\(^2\)/s/Hz/sr (ergs/cm\(^2\)/s/Hz/sr)

Function of position and direction

Independent of distance when no sources or sinks

\( s \) is normal to \( dA \)
MC packet moving in direction $\theta$, contributes to the specific intensity:

\[
I_\nu = \frac{dE_\nu}{\cos \theta dA dt d\nu d\Omega}
\]

\[
\Delta I_\nu = \frac{E_i}{\cos \theta \Delta A \Delta t \Delta \nu \Delta \Omega}
\]
$I_\nu$ is a *distribution function*. MC works with *discrete* energies. Binning the packets into directions, frequencies, etc, enables us to simulate a distribution function:

- Spectrum: bin in frequency
- Scattering phase function: bin in angle
- Images: bin in spatial location
Photon Interactions

Beam of radiation passes through the slab

Volume = $A \, dl$

Number density $n$ (units Length$^{-3}$)

Cross section $\sigma$ (units Length$^2$)

Energy removed from beam per particle /$t$ / $\nu$ / d$\Omega$ = $I_\nu \, \sigma$
Intensity differential over \( dl \) is \( dI_\nu = - I_\nu \ n \ \sigma \ dl \). Therefore

\[
I_\nu (l) = I_\nu (0) \ \exp(-n \ \sigma \ l)
\]

Fraction scattered or absorbed / length = \( n \ \sigma \)

\( n \ \sigma \) = volume absorption coefficient = \( \rho \ \kappa \)

Mean free path = \( 1/(n \ \sigma) \) = average dist between interactions

Probability of interaction over \( dl \) is \( n \ \sigma \ dl \)

Probability of traveling \( dl \) without interaction is \( 1 - n \ \sigma \ dl \)

\[ \text{Probability of traveling } L \text{ before interacting is } \]

\[ P(L) = (1 - n \ \sigma \ L / N) \ (1 - n \ \sigma \ L / N) \ldots \]

\[ = (1 - n \ \sigma \ L / N)^N = \exp(-n \ \sigma \ L) \] (as \( N \rightarrow \text{infty} \))

\[ P(L) = \exp(-\tau) \]

\( \tau \) = number of mean free paths over distance \( L \)
Cross sections vary with the energy of the photons or particles. Can be continuous, lines, edges.

**Astronomy**: dust more opaque at short wavelengths.
Atmospheric physics: water in clouds, atmospheric opacity

Solar Radiation Spectrum

Optical absorption opacity of water

Infrared absorption opacity of:
- Liquid water
- Ice
- Atmospheric water vapour
Medical physics: blood, water, molecules – therapeutic window
Fluence rate (mean intensity) in skin model

Fluence Rate $\psi/\psi_0$ vs. Depth [mm]

- 405 nm
- 630 nm

Louise Campbell et al. (2014)
Photodynamic Therapy: Photodynamic Dose

Louise Campbell et al. (2014)
Fission cross sections for slow and fast neutrons
Resonances correspond to energy levels in the nucleus
PDF for photons to travel $\tau$ before an interaction is $\exp(-\tau)$. If we pick $\tau$ uniformly over the range 0 to infinity we will not reproduce $\exp(-\tau)$. Want to pick lots of small $\tau$ and fewer large $\tau$. Same with a scattering phase function: want to get the correct number of packets scattered into different directions, forward and back scattering, etc.
Cumulative Distribution Function

\[ \text{CDF} = \text{Area under PDF} = \int P(x) \, dx \]

Randomly choose \( \tau, \theta, \lambda, \ldots \) so that PDF is reproduced

\( \xi \) is a random number uniformly chosen in range (0,1)

\[ \xi = \int_{a}^{b} P(x) \, dx \Rightarrow X \]

This is the \textit{fundamental principle} behind Monte Carlo techniques and is used to sample randomly from PDFs.
e.g., \( P(\theta) = \cos \theta \) and we want to map \( \xi \) to \( \theta \). Choose random \( \theta \)s to “fill in” \( P(\theta) \).

\[
\xi_i = \int_0^{\theta_i} P(\theta) \, d\theta = \sin \theta_i \Rightarrow \theta_i = \sin^{-1} \xi_i
\]

Sample many random \( \theta_i \) in this way and “bin” them, we will reproduce the curve \( P(\theta) = \cos \theta \).
Choosing a Random Optical Depth

\[ P(\tau) = \exp(-\tau), \text{ i.e., packet travels } \tau \text{ before interaction} \]

\[ \xi = \int_{0}^{\tau} e^{-\tau'} \, d\tau' = 1 - e^{-\tau} \Rightarrow \tau = -\log(1 - \xi) \]

Since \( \xi \) is in range (0,1), then (1-\( \xi \)) is also in range (0,1), so we may write:

\[ \tau = -\log \xi \]

Physical distance, \( L \), that the packet has traveled from:

\[ \tau = \int_{0}^{L} n \sigma \, ds \]
Random Isotropic Direction

Solid angle is $d\Omega = \sin \theta \, d\theta \, d\phi$, choose $(\theta, \phi)$ so they fill in PDFs for $\theta$ and $\phi$. $P(\theta)$ normalized over $[0, \pi]$, $P(\phi)$ normalized over $[0, 2\pi]$:

$$P(\theta) = \frac{1}{2} \sin \theta \quad P(\phi) = \frac{1}{2\pi}$$

Using fundamental principle from above:

$$\xi = \int_0^\theta P(\theta) \, d\theta = \frac{1}{2} \int_0^\theta \sin \theta \, d\theta = \frac{1}{2} (1 - \cos \theta)$$

$$\xi = \int_0^\phi P(\phi) \, d\phi = \frac{1}{2\pi} \int_0^\phi d\phi = \frac{\phi}{2\pi}$$

Use this for emitting packets isotropically from a point source, or choosing isotropic scattering direction. See later about computational cost of functions!!
Rejection Method

Used when we cannot invert the PDF as in the above examples to obtain analytic formulae for random $\theta$, $\lambda$, etc.

### e.g., $P(x)$ can be complex function or tabulated

Multiply two random numbers:
uniform probability / area

Pick $x_1$ in range $[a, b]$: $x_1 = a + \xi(b - a)$, calculate $P(x_1)$
Pick $y_1$ in range $[0, P_{\text{max}}]$: $y_1 = \xi P_{\text{max}}$
If $y_1 > P(x_1)$, reject $x_1$. Pick $x_2, y_2$ until $y_2 < P(x_2)$: accept $x_2$
Efficiency = Area under $P(x)$
Albedo
Packet gets to interaction location at randomly chosen $\tau$, then decide whether it is scattered or absorbed. Use the **albedo** or **scattering probability**. Ratio of scattering to total opacity:

\[
a = \frac{\sigma_S}{\sigma_S + \sigma_A}
\]

To decide if a packet is scattered: pick a random number in range (0, 1) and scatter if $\xi < a$, otherwise packet absorbed

Now have the tools required to write a Monte Carlo radiation transfer program for isotropic scattering in a constant density slab or sphere