MCRT: L5

- MCRT estimators for mean intensity, absorbed radiation, radiation pressure, etc
- Path length sampling: mean intensity, fluence rate, number of absorptions
- Random number generators
Mean Intensity

\[ J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi I_\nu \sin \theta \, d\theta \, d\phi \]

Same units as \( I_\nu \): J/m\(^2\)/s/Hz/sr (ergs/cm\(^2\)/s/Hz/sr)

Function of position (and time), but not direction

Determines heating, ionization, level populations, etc
What is \( J \) at \( r \) from a star with uniform specific intensity \( I_* \) across its surface?

\[
I = I_* \quad \text{for} \quad 0 < \theta < \theta_* \quad (\mu_* < \mu < 1); \quad \mu = \cos \theta
\]

\[
I = 0 \quad \text{for} \quad \theta > \theta_* \quad (\mu < \mu_*)
\]

\[
J = \frac{1}{2} \int_{\mu_*}^{1} I \, d\mu = \frac{1}{2} I_* (1 - \mu_*)
\]

\[
J = I_* \left( \frac{1}{2} \left( 1 - \sqrt{1 - \frac{R_*^2}{r^2}} \right) \right) = w I_*
\]

\( w = \text{dilution factor} \)

Large \( r \), \( w = R^2/4r^2 \)
Monochromatic Flux

Energy passing through a surface. Units: J/s/m²/Hz

\[ dF_\nu = I_\nu \cos \theta \, d\Omega \]

\[
F_\nu = \int I_\nu \cos \theta \, d\Omega = \int_0^{2\pi} \int_0^\pi I_\nu \cos \theta \sin \theta \, d\theta \, d\phi
\]

If \( I_\nu \) is isotropic thermal radiation, get:

\[
F_\nu = \pi I_\nu = \pi B_\nu
\]
Momentum Flux

The momentum of a photon is $E/c$

Momentum flux $p_\nu$ in the direction of $n =$ photon flux times momentum per photon:

$$p_\nu (n) = 1/c \int I_\nu \cos^2 \theta \, d\Omega$$

One factor of $\cos \theta$ comes from foreshortening of $dA$

Only the normal component of the momentum acts on the surface, hence the second factor of $\cos \theta$

Units: N/m$^2$/Hz
Energy density of radiation

Consider a cylinder along a ray of length \( c \, dt \). Define:

\[ u_{\nu}(\Omega) = \text{energy per unit solid angle per unit volume per unit frequency in the cylinder:} \]

\[ dE = u_{\nu}(\Omega) \, dV \, d\Omega \, dv = u_{\nu}(\Omega) \, (dA \times c \, dt) \, d\Omega \, dv \]

All this radiation will exit the cylinder through \( dA \) in time \( dt \), so:

\[ dE = I_{\nu} \, dA \, d\Omega \, dt \, dv \]

Equating gives:

\[ u_{\nu}(\Omega) = I_{\nu} / c \]

Integrating over angles, we obtain the specific energy density, \( u_{\nu} \) (units \( \text{J/m}^3/\text{Hz} \)). This is the energy per unit volume per unit frequency interval,

\[ u_{\nu} = \int u_{\nu}(\Omega) \, d\Omega = (1 / c) \int I_{\nu} \, d\Omega = (4\pi / c) \, J_{\nu} \]

The total energy density of radiation requires one more integration over frequencies (this has dimensions of Energy / Volume):

\[ u = \int u_{\nu} \, dv = (4 \pi / c) \int J_{\nu} \, dv \]
Moments of the Radiation Field

First three moments of specific intensity are named $J$ (zeroth moment), $H$ (first), and $K$ (second):

\[
J_\nu = \frac{1}{4\pi} \int I_\nu \ d\Omega
\]

\[
H_\nu = \frac{1}{4\pi} \int I_\nu \cos \theta \ d\Omega
\]

\[
K_\nu = \frac{1}{4\pi} \int I_\nu \cos^2 \theta \ d\Omega
\]

Physically: $J = \text{mean intensity;} \ H = F / 4\pi$

$K$ related to momentum flux:

\[
p_\nu = \frac{4\pi}{c} K_\nu
\]
Intensity Moments

The moments of the radiation field are:

\[ J_v = \frac{1}{4\pi} \int I_v \, d\Omega \quad H_v = \frac{1}{4\pi} \int I_v \, \mu \, d\Omega \quad K_v = \frac{1}{4\pi} \int I_v \, \mu^2 \, d\Omega \]

\( J \) – mean intensity; \( H \) – flux; \( K \) – momentum flux

Compute these moments throughout the slab. First split the slab into layers, then tally number of packets, weighted by powers of their direction cosines to obtain \( J, H, K \). Contribution to specific intensity from a single packet is:

\[ \Delta I_v = \frac{\Delta E}{|\mu|\Delta A \Delta t \Delta v \Delta \Omega} = \frac{F_v}{|\mu|N \Delta \Omega} = \frac{\pi B_v}{|\mu|N \Delta \Omega} \]
Substitute into intensity moment equations and convert the integral to a summation to get:

\[ J_\nu = \frac{B_\nu}{4N} \sum \frac{1}{|\mu_i|} \quad H_\nu = \frac{B_\nu}{4N} \sum \frac{\mu_i}{|\mu_i|} \quad K_\nu = \frac{B_\nu}{4N} \sum \frac{\mu_i^2}{|\mu_i|} \]

Note the mean flux, \( H \), is just the net energy passing each level: number of packets traveling up minus number traveling down.

Pathlength formula (Lucy 1999)
Long history of use in neutronics

\[ J_i = \frac{L}{4\pi N \Delta V_i} \sum l \]
Some Monte Carlo photon packets may pass through a cell without interacting (scatter or absorbed), but the path length estimator ensures they still contribute to the estimates for mean intensity, absorbed energy, radiation pressure, etc.

\[ J_i = \frac{L}{4\pi N \Delta V_i} \sum l \]
Summing path lengths gives better estimates for intensities, absorbed energy, radiation pressure, etc. More packets pass through a cell than interact with a cell.

Mean intensity, $J$, related to photon energy density, $u$, via

$$u_{\nu} = 4\pi J_{\nu} / c$$

Packet contributes a fraction $\varepsilon_{\nu} t/\Delta t$ to the energy density of a cell where $t = l/c$ is time the packet spends in a cell, so can form Monte Carlo estimator:

$$u_{\nu} = \frac{1}{c \Delta t \Delta V_i} \sum \varepsilon_{\nu} l$$

Where $\varepsilon_{\nu} = $ MC packet energy $= L \Delta t / N$. Hence, get estimator for $J$ which will be accurate in optically thin regions:

$$J_i = \frac{L}{4\pi N \Delta V_i} \sum l$$
How much energy absorbed in a cell? Could count number of absorption events in each cell, but this is inaccurate for optically thin systems. We know the change in intensity for radiation passing through a medium with absorbing particles is

\[ \text{d}I = - I n \sigma_{\text{abs}} \text{d}l = - I \text{d}\tau_{\text{abs}} \]

Hence, a Monte Carlo estimator for absorbed energy:

\[ E_{i}^{\text{abs}} = \frac{L}{4\pi N \Delta V_i} \sum n \sigma_{\text{abs}} l \]
Random Number Generators

• Want random numbers in range $0 < \xi < 1$
• Generate sequence of numbers rapidly
• No patterns or correlations
• Pass statistical tests for randomness
• Because using computer algorithms the sequence will be periodic, so period should be as long as possible – pseudo random numbers

Anyone who considers arithmetical methods for producing random digits is, of course, in a state of sin.

John von Neumann
Random Number Generators

- Pseudo random number generators PRNGs
- $\xi = \text{ran}(\text{seed})$, seed usually an integer, updated in each call to ran
- Can easily de-bug Monte Carlo codes: if use same seed for PRNG will get same random sequence
- Use uniformly distributed random numbers from $(0 < \xi < 1)$ to sample from complex functions
- Much research devoted to PRNGs
- Computer modular arithmetic, remainders and bits
Middle Square Method

• Used by von Neumann in 1940s
• Recursive relation: square a $n$-digit number and take the middle $n$ digits (add zeroes to make a $2n$ digit number if necessary), repeat process

• e.g., $n = 4$: $2568^2 = 06594624$ gives 5946, $\xi = 0.5946$
  $5946^2 = 35354916$ gives 3549, $\xi = 0.3549$
  $3549^2 = 12595401$ gives 5954, $\xi = 0.5954$
  etc, etc

• Problems: short period, can get stuck in very short loops, or crash (e.g., 0000)

• Von Neumann acknowledged problems, but found this fast (1940s), adequate for problems, and crashes were obvious
Linear Congruential Generators

- Based on integer recurrence relation:
  \[ y_{i+1} = \text{mod}(a y_i + c, M) \]

- mod(A,B) gives remainder when A divided by B

- Careful choice of \(a, c, M\) gives periods around \(2^{32} (10^9)\)

- Random number in range \((0,1)\) from: \(\xi_i = \frac{y_i}{M}\)

- ran2.f from *Numerical Recipes* is more complex, involving shuffling of sequences. Authors offered prize of $1000 if anyone finds a statistical test that ran2 fails.
Good and Bad Random Number Generators

\[ y_{i+1} = \text{mod}(a y_i + c, M) \]
\[ \xi_i = \frac{y_i}{M} \]

- \( a = 1366, c = 150889, M = 714025 \)
- \( a = 137, c = 187, M = 256 \)
- “First return maps” – plot successive pairs of \((\xi_i, \xi_{i+1})\)

![Figure 14.4. First return maps for (a) Random Number Generator #1; (b) Random Number Generator #2.](image-url)
Should we worry…?

- Maybe… but if you read you’ll get scared!
- OK for our needs, need better for cryptography
- Mersenne Twister: period $2^{19937}-1 \sim 4.3 \times 10^{601}$