In the first lecture, we introduced the flux $\boldsymbol{F_\nu}$, specific intensity $\boldsymbol{I_\nu}$, and momentum flux $\boldsymbol{p_\nu}$, which are all monochromatic quantities (units include Hz^{-1})

Can also define their frequency integrated quantities:

$$F = \int F_\nu \, d\nu$$
$$I = \int I_\nu \, d\nu$$
$$p = \int p_\nu \, d\nu$$

We can also define the **mean intensity**, $\boldsymbol{J_\nu}$:

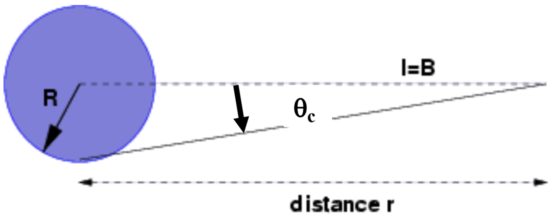
$$J_\nu = 1/4\pi \int I_\nu \, d\Omega$$

$\boldsymbol{J_\nu}$ = specific intensity averaged over all solid angles $\boldsymbol{\Omega}$

Inverse square law

Check: does constant specific intensity along a ray yield the inverse square law?

Consider a spherical source of uniform intensity B. At a point outside the sphere, $I=B$ if the ray intersects the sphere and $I=0$ otherwise. Integrate over the visible area of the sphere to find the flux,



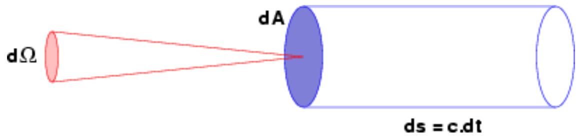
$$F = \int I \cos\theta \, d\Omega = B \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin\theta \, \cos\theta \, d\theta$$

where the upper limit on θ integral is where a ray just grazes the sphere, $\sin\theta_c = R/r$. The integral gives,

$$F = \pi B \, (1 - \cos^2\theta_c) = \pi B \sin^2\theta_c = \pi B \, (R/r)^2$$

i.e., it all works. Specific intensity is constant but the solid angle drops with radius to give the inverse square law. Setting $r = R$, the flux at the surface of an object of uniform brightness B is $\boldsymbol{F = \pi B}$.

Energy density of radiation



Consider a cylinder along a ray of length *c dt*. Define:

$u_\nu(\Omega)$ = energy per unit solid angle per unit volume per unit frequency in the cylinder:

$$dE = u_\nu(\Omega)\, dV\, d\Omega\, d\nu = u_\nu(\Omega)\, (dA\ c\ dt)\, d\Omega\, d\nu$$

All this radiation will exit the cylinder through dA in time dt, so:

$$dE = I_\nu\, dA\, d\Omega\, dt\, d\nu$$

Equating the above gives: $u_\nu(\Omega) = I_\nu / c$

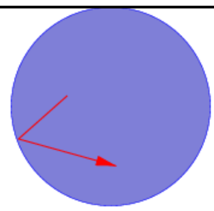
Integrating over angles, we obtain the **specific energy density**, u_ν. This is the energy per unit volume per unit frequency interval,

$$u_\nu = \int u_\nu(\Omega)\, d\,\Omega = (1/c) \int I_\nu\, d\Omega = (4\pi/c)\, J_\nu$$

As before, the total energy density of radiation requires one more integration over frequencies and has dimensions of Energy / Volume,

$$u = \int u_\nu\, d\nu = (4\,\pi/c) \int J_\nu\, d\nu$$

Radiation pressure



Consider an enclosure containing an isotropic radiation field. The pressure of radiation on the wall is given by,

$$p_\nu = (2/c) \int I_\nu \cos^2\theta\, d\Omega$$

except that now (1) we have a factor of 2 because each photon exerts twice its normal component of momentum on reflecting from the wall, and (2) the integral is only over $2\,\pi$ (one hemisphere).

By definition, an isotropic radiation field has no angular dependence, so $I_\nu = J_\nu$. The total radiation pressure is then,

$$p = (2/c) \int J_\nu\, d\nu \int \cos^2\theta\, d\Omega$$

Doing the integral over solid angle gives (Tut sheet 1, Q1: do this integral),

$$p = (4\,\pi/3\,c) \int J_\nu\, d\nu$$

$$p = (1/3)\, u$$

Result: The radiation pressure of an isotropic radiation field is one third of the energy density of the radiation.

Radiative transfer equation

Earlier, we defined the specific intensity and showed that in free space it was constant along a ray

$$dI_\nu/ds = 0$$

We now consider a medium in which there may be *emission* and/or *absorption*.

Emission

Define the spontaneous **emission coefficient** ***j***. This is the energy emitted per unit time per unit solid angle and per unit volume,

$$dE = j\, dV\, d\Omega\, dt$$

As usual, we can also define a monochromatic version which is per unit frequency,

$$dE = j_\nu\, dV\, d\Omega\, dt\, d\nu$$

which has units of W m^{-3} Hz^{-1} sr^{-1} **.** In going a distance *ds*, a beam of radiation sweeps out a volume $dV = dA\, ds$, where *dA* is the area. The change in the intensity due to spontaneous emission is,

$$dI_\nu = j_\nu\, ds$$

Note: we restrict ourselves to *spontaneous* emission. There can also be *stimulated* emission, but this depends upon I_ν and is more conveniently treated as "negative absorption."

Related quantities

The emission coefficient can vary with angle (i.e., more emission in some directions than others). Simplifications are possible for an isotropic emitter. Then,

$$j_\nu = \frac{1}{4\pi} P_\nu$$

where P_ν is the emitted power per unit volume per unit frequency.

Another related quantity is the **emissivity** ε_ν, defined as the power emitted spontaneously per unit mass per unit frequency. The emissivity is integrated over all angles. For an isotropic source,

$$dE = \varepsilon_\nu dm\; dt\; d\nu \frac{d\Omega}{4\pi}$$

The last factor is the fraction of energy radiated into $d\Omega$. Using $dm = \rho\, dV$, where ρ is the density we find,

$$dE = \varepsilon_\nu dm\; dt\; d\nu \frac{d\Omega}{4\pi} = j_\nu\; dV\; dt\; d\nu\; d\Omega$$

and hence the relation between the emissivity and the emission coefficient is

$$j_\nu = \frac{\epsilon_\nu\, \rho}{4\pi}$$

Lecture 2 revision quiz

- Suppose a star radiates isotropically with specific intensity I_ν
  - What is the mean intensity J_ν at the surface of the star?
  - What is the mean intensity at distance a from the centre of a star of radius R?
  - Sanity check: is your answer to the second part valid at the surface, where $a = R$?
- Integrate $\int d\Omega$, $\int \cos\theta \, d\Omega$ and $\int \cos^2\theta \, d\Omega$ to verify that there are no mistakes in the lecture notes. Which radiant quantity do you associate with each of these 3 integrals?
- Fill in the steps in the derivation to derive the inverse square law.
- For an isotropic radiation field compute the flux and mean intensity, and show that the radiation pressure $p = u/3$
- What's the difference between emission coefficient j_ν and emissivity ε_ν ?