Thermal Radiation

Now we consider **thermal radiation** emitted by matter in thermal equilibrium. If the medium is also optically thick, it becomes **blackbody radiation**.

An extremely good example:
The cosmic microwave background spectrum,
with $T_{CMB} = 2.729 \pm 0.004$ K.

(COBE, Fixsen et al., 1996, ApJ, 473, 576)

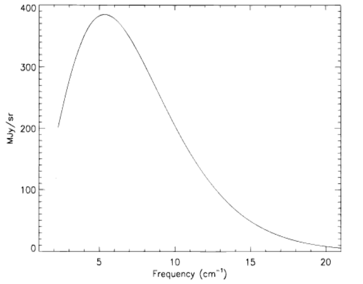


FIG. 4.—Uniform spectrum and fit to Planck blackbody (T). Uncertainties are a small fraction of the line thickness.

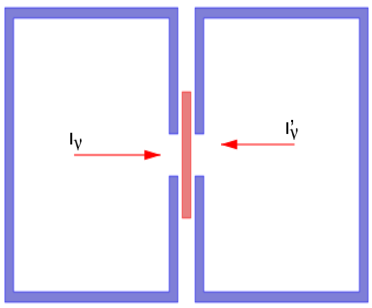
In other cases, such as stars (where we know better), or accretion discs (where we don't), blackbody radiation provides a useful first approximation to the thermal spectrum.

For blackbody radiation, the energy density, flux and intensity depend on temperature only.

Blackbody Radiation

Consider insulated cavity at uniform temperature *T*. Photons created and destroyed until radiation is in thermal equilibrium with the cavity. Can sample the radiation through a small hole.

Specific intensity of bb radiation is a function of *T* only. Proof:



- Imagine two cavities at same *T*, containing radiation with intensity I_ν and I_ν'
- Connect cavities with a narrow-band filter that passes radiation only around frequency ν.
- If I_ν is not equal to I'_ν radiation will flow in one direction between the two cavities. Violates 2nd law of thermodynamics. Hence, $I_\nu = I'_\nu$

Can show that I_ν is isotropic and independent of the shape of the cavity. It is a function only of *T* (and ν, of course).

$$I_\nu \text{ (black body radiation)} = B_\nu (T)$$

$B_\nu(T)$ is called the Planck function.

Kirchhoff's Law

Relates the emission coefficient j_ν to the absorption coefficient α_ν for thermal emission.

Imagine placing a thermally emitting material with source function S_ν at temperature T inside a blackbody cavity of the same T. Intensity of blackbody radiation does not depend on cavity shape, so the intensity of radiation must be unchanged. Thus, the source function of the material must equal the intensity of blackbody radiation:

$$S_\nu = B_\nu(T)$$

$$\Rightarrow j_\nu = \alpha_\nu B_\nu(T)$$

This is Kirchhoff's Law. Note the distinction:

Thermal radiation for which $S_\nu = B_\nu$

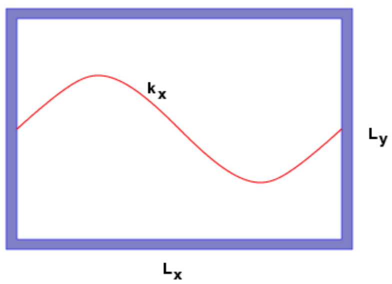
Blackbody radiation for which $I_\nu = B_\nu$

Blackbody radiation is a special case of thermal radiation for optically thick media. Blackbody radiation is homogenous and isotropic, so

$$p = \frac{1}{3}u$$

Planck Spectrum

To derive the Planck spectrum, we again consider a blackbody cavity of dimensions L_x, L_y, L_z.



To find the energy density of the radiation in the cavity, need to know:

- Number of modes (or states) of the EM field inside the cavity.
- Average energy of each mode

Photon of frequency ν has wave vector $\boldsymbol{k} = (2\pi/\lambda)\boldsymbol{n} = (2\pi\ \nu/c)\boldsymbol{n}$

For $\lambda \ll L$, treat the photon as a standing wave. Permitted states in the x direction say have an integer number of nodes n_x, where:

$$n_x = \frac{L_x}{\lambda} = \frac{k_x L_x}{2\pi}$$

For large n_x, the number of allowed states in a wavenumber interval Δk_x is,

$$\Delta n_x = \frac{L_x \Delta k_x}{2\pi}$$

$d^3k = \Delta k_x \Delta k_y \Delta k_z$ the number of states is,

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = \frac{L_x L_y L_z d^3k}{(2\pi)^3}$$

Since $L_x L_y L_z = V$, the **number of states per unit volume per unit three-dimensional wavevector is 2 / (2 π)³**, where the extra factor of 2 accounts for photons having two polarization states.

Magnitude of wavevector $k^2 = k_x^2 + k_y^2 + k_z^2$ (cf. radius of sphere in k-space).

Modes with wavenumbers in range k to $k+dk$ occupy shell of volume $4\pi k^2\,dk$, or just $k^2\,dk$ per unit solid angle (because black body radiation is isotropic).

Multiply by $2/(2\pi)^3$ to get **number of states per unit volume per unit frequency per unit solid angle**:

$$\frac{2}{(2\pi)^3} d^3k = \frac{2}{(2\pi)^3} k^2 dk = \frac{2}{(2\pi)^3} \frac{(2\pi)^3 \nu^2}{c^3} d\nu = \frac{2\nu^2}{c^3} d\nu$$

$\rho_s = 2\nu^2/c^3$ is the *density of states*. ρ_s increases without limit at high frequency. We recreate a famous error of classical physics if we assume that all states are occupied. Instead, we ask what is the average energy per state?

Photon of frequency ν has energy $h\nu$, where h is Planck's constant. Each state can contain n photons, where $n = 0,1,2,\ldots$ The total energy per state is then, $E_n = nh\nu$

Statistical mechanics: probability of a state having energy E_n is proportional to $e^{-E_n/kT}$ where k is Boltzmann's constant. Define $\beta = 1/kT$.

Weighted average energy $\bar{E}$ is then,

$$\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln\left(\sum_{n=0}^{\infty} e^{-\beta E_n}\right)$$

The bracket is a geometric series with sum, $\sum_{n=0}^{\infty} e^{-nh\nu\beta} = \left(1 - e^{-\beta h\nu}\right)^{-1}$

so we have, $$\bar{E} = \frac{h\nu}{\exp(h\nu/kT) - 1}$$

i.e., the average occupancy of a state of frequency ν is

$$n_\nu = [\exp(h\nu/kT) - 1]^{-1}$$

The energy per unit volume per unit frequency interval per unit solid angle is then the product of the density of states ρ_s and the average energy $\bar{E}$ per state. By definition this is $u_\nu(\Omega)$:

$$u_\nu(\Omega)dVd\nu d\Omega = \left(\frac{2\nu^2}{c^3}\right)\frac{h\nu}{\exp(h\nu/kT)-1}dVd\nu d\Omega$$

$$\Rightarrow u_\nu\,(\Omega) = \frac{2h\nu^3/c^3}{\exp(h\nu/kT)-1}$$

Earlier we found that, $u_\nu(\Omega) = I_\nu/c$

For blackbody radiation, where $I_\nu = B_\nu$, we finally obtain,

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT)-1}$$

This is the **Planck law**.

We can also write the Planck law per unit wavelength instead of per unit frequency,

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{exp(hc/\lambda kT)-1}$$

B_ν and B_λ do not peak at the same place, because ν and λ are not linearly related. This is a common source of confusion.

Properties of the Planck Spectrum

- Low frequency limit $h\nu \ll kT$:

  Rayleigh-Jeans law

  $$I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2}kT$$

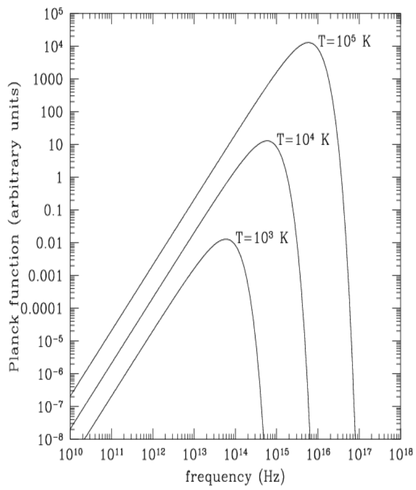
  The classical result. Useful at low frequencies, especially in the radio part of the spectrum.

- High frequency limit $h\nu \gg kT$:

  Wien Law

  $$I_\nu^W(T) = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right)$$

  i.e., the spectrum cuts off exponentially at high frequency.



- Increasing T increases $B_\nu(T)$ at all frequencies
- Wien displacement law. The peak of $B_\nu(T)$ increases linearly with T,

  $h\nu_{max} = 2.82\, kT \Rightarrow \nu_{max} / T = 5.88 \times 10^{10}$ Hz K^{-1}

  NB c/ν_{max} is **not** the wavelength at which $B_\lambda(T)$ peaks!!!

- Stefan-Boltzmann law: Integrate B_ν over frequency to obtain:
- Energy density of blackbody radiation is $u(T) = aT^4$

  where a is the radiation constant: $$a = \frac{8\pi^5 k^4}{15c^3h^3} = 7.56\times10^{-16}\ \mathrm{J\ m^{-3}K^{-4}}$$

- Flux from an isotropically emitting blackbody surface is $F = \sigma T^4$

  where σ is the Stefan-Boltzmann constant:

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} = 5.67\times10^{-8}\ \mathrm{J\ m^{-2}K^{-4}s^{-1}}$$

(NB flux from an isotropic emitting surface is $\pi \times$ specific intensity)

Lecture 4 revision quiz

- What is the volume of a spherical shell of radius k and thickness dk?
- Sanity check: differentiate the right-hand side of equation 9 to show that:

$$\frac{\partial}{\partial\beta}\ln\left(\sum_{n=0}^{\infty} e^{-\beta E_n}\right) = -\frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}$$

- Sanity check: find a reference to help you verify:

$$\sum_{n=0}^{\infty} e^{-nh\nu\beta} = \left(1 - e^{-\beta h\nu}\right)^{-1}$$

- Given that the frequency integrated Planck function equals $\sigma T^4/\pi$, show that for black body radiation $u = a\, T^4$, where $a = 4\sigma/c$.