Relations between the Einstein coefficients

- In thermodynamic equilibrium, transition rate (per unit time per unit volume) from level 1 to level 2 must equal transition rate from level 2 to level 1.
- If the number density of atoms in level 1 is n_1, and that in level 2 is n_2, then

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

$$\Rightarrow \bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

Compare mean intensity with Planck function

- Use Boltzmann's law to obtain the relative populations n_1 and n_2 in levels with energies E_1 and E_2:

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21})\exp(h\nu/kT) - 1}$$

- In TE, mean intensity equals the Planck function, $\bar{J} = B_\nu$

where $$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$$

Einstein relations

- To make mean intensity = Planck function, Einstein coefficients must satisfy the *Einstein relations*,

$$g_1 B_{12} = g_2 B_{21} \qquad A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

- The Einstein relations:
  - Connect *properties of the atom*. Must hold even out of thermodynamic equilibrium
  - Are examples of *detailed balance relations* connecting absorption and emission
  - Allow determination of all the coefficients given the value of one of them

- We can write the emission and absorption coefficients j_ν, α_ν etc in terms of the Einstein coefficients

Emission coefficient

- Assume that the frequency dependence of radiation from spontaneous emission is the same as the line profile function $\phi(\nu)$ governing absorption
- There are n_2 atoms per unit volume
- Each transition gives a photon of energy $h\nu_0$, which is emitted into 4π steradians of solid angle
- Energy emitted from volume dV in time dt, into solid angle $d\Omega$ and frequency range $d\nu$ is then:

$$dE = j_\nu dV d\Omega dt d\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu) dV d\Omega dt d\nu$$

$\Rightarrow$ Emission coefficient, $\quad j_\nu = \dfrac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)$

Absorption coefficient

- Likewise, we can write the absorption coefficient:

$$\alpha_\nu = \frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21})\phi(\nu)$$

- This includes the effects of stimulated emission as "negative absorption"

Radiative transfer again

- The transfer equation $\dfrac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$

  becomes:

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21})\phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21}\phi(\nu)$$

- Substituting for the Einstein relations, the source function and the absorption coefficient are,

$$S_\nu = \frac{2h\nu^3}{c^2}\left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1} \qquad \alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}\left(1 - \frac{g_1 n_2}{g_2 n_1}\right)\phi(\nu)$$

Populations of states

- Non-thermal emission when: $\frac{n_2}{n_1} \neq \frac{g_2}{g_1} e^{-h\nu/kT}$
- Populations of different energy levels depend on detailed processes that populate/depopulate them.
- In thermal equilibrium it's easy – Boltzmann gives relative populations, otherwise hard
- Population of a level with energy E_i above ground state and statistical weight g_i is:

$$N_i = \frac{N}{U} g_i e^{-E_i/kT}$$

- N is the total number of atoms in all states per unit volume and U is the *partition function*:

$$N = \sum_i N_i \Rightarrow U = \sum_i g_i e^{-E_i/kT}$$

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- At low T only the first term is significant so U = statistical weight g_1 of ground state.
- Beware: At finite T, g_i for higher states becomes large while Boltzmann factor $\exp(-E_i / kT)$ tends to a constant once E_i approaches ionization energy.
  - Partition function sum diverges
  - Idealized model of isolated atom breaks down due to loosely bound electrons interacting with neighbouring atoms
  - Solution: cut off partition function sum at finite n, e.g. when Bohr orbit radius equals interatomic distance:

$$a_0 \approx 5\times10^{-11} Z^{-1} n^2 \text{ m} \approx N^{-1/3}$$

  - More realistic treatments must include plasma effects. In practice: don't worry too much about how exactly to cut it off

Masers (bound-bound)

- In thermal equilibrium, the excited states of an atom are less populated

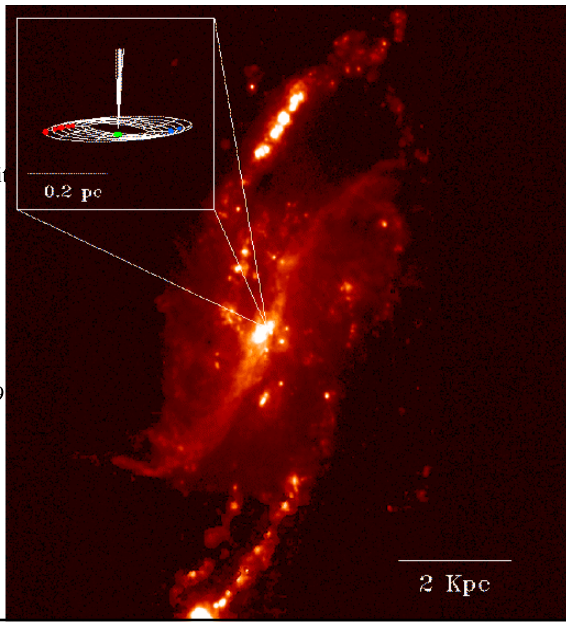
$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT} < 1 \quad \text{and,} \quad \frac{N_1}{g_1} > \frac{N_2}{g_2}$$

- If some mechanism can put enough atoms into an upper state the normal population of the energy levels is turned into an inverted population,
- This leads to: $\frac{N_1}{g_1} < \frac{N_2}{g_2}$
- A negative absorption coefficient – amplification!
- At microwave frequencies, astrophysical masers typically involve H_2O or OH
  - produce highly polarized radiation (Zeeman effect producing circular polarization)
  - Have extremely high brightness temperatures (all radiation emitted in a narrow line)

Masers in NGC4258

Water vapour masers have been observed in the inner pc of the galaxy NGC4258

- Velocities trace Keplerian motion around a central mass
- Strongest evidence for a black hole wi mass 4×10^7 $M_\odot$
- Measurement of proper motions provides geometric distance to the galaxy and estimate of the Hubble constant
- Masers also seen in star forming regions
- Herrnstein et al 1999, *Nature* 400, 539



Lecture 7 revision quiz

- Write down the equation balancing upward and downward radiative transition rates for a 2-level atom in a radiation field of mean intensity *J.* Sketch the transitions that are balanced in the equation.
- Use Boltzmann's law to fill in the step in the calculation between slide 1 and slide 2
- What do the Einstein coefficients A_{21}, B_{21} and B_{12} symbolise?
- What are their units?
- Why is there no A_{12} coefficient?
- What is the use of the Partition function *U*?