Relations between the Einstein coefficients

- Additional reading: Böhm-Vitense Ch 13.1, 13.2
- In thermodynamic equilibrium, transition rate (per unit time per unit volume) from level 1 to level 2 must equal transition rate from level 2 to level 1.
- If the number density of atoms in level 1 is $n_1$, and that in level 2 is $n_2$, then

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

$$\Rightarrow \bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$
Compare mean intensity with Planck function

- Use Boltzmann’s law to obtain the relative populations $n_1$ and $n_2$ in levels with energies $E_1$ and $E_2$:

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu/kT) - 1}$$

- In TE, mean intensity equals the Planck function, $\bar{J} = B_\nu$

where $B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$
Einstein relations

- To make mean intensity = Planck function, Einstein coefficients must satisfy the *Einstein relations*,

\[ g_{12} B_{12} = g_{21} B_{21} \quad A_{21} = \frac{2h\nu^3}{c^2} B_{21} \]

- The Einstein relations:
  - Connect *properties of the atom*. Must hold even out of thermodynamic equilibrium.
  - Are examples of *detailed balance relations* connecting absorption and emission.
  - Allow determination of all the coefficients given the value of one of them.

- We can write the emission and absorption coefficients \( j_\nu, \alpha_\nu \) etc in terms of the Einstein coefficients.
Emission coefficient

• Assume that the frequency dependence of radiation from spontaneous emission is the same as the line profile function $\phi (\nu)$ governing absorption.

• There are $n_2$ atoms per unit volume.

• Each transition gives a photon of energy $h\nu_0$, which is emitted into $4\pi$ steradians of solid angle.

• Energy emitted from volume $dV$ in time $dt$, into solid angle $d\Omega$ and frequency range $d\nu$ is then:

$$dE = j_\nu dV d\Omega dt d\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu) dV d\Omega dt d\nu$$

⇒ Emission coefficient, \[ j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu) \]
Absorption coefficient

• Likewise, we can write the absorption coefficient:

\[
\alpha_{\nu} = \frac{h\nu}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi(\nu)
\]

• This includes the effects of stimulated emission.
Radiative transfer again

- The transfer equation \( \frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \)

becomes:

\[
\frac{dI_\nu}{ds} = -\frac{hv}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi(\nu) I_\nu + \frac{hv}{4\pi} n_2 A_{21} \phi(\nu)
\]

- Substituting for the Einstein relations, the source function and the absorption coefficient are,

\[
S_\nu = \frac{2hv^3}{c^2} \left( \frac{g_2 n_1}{g_1 n_2} - 1 \right)^{-1} \quad \alpha_\nu = \frac{hv}{4\pi} n_1 B_{12} \left( 1 - \frac{g_1 n_2}{g_2 n_1} \right) \phi(\nu)
\]
Non-thermal emission

- All cases where: \( \frac{n_2}{n_1} \neq \frac{g_2}{g_1} e^{-h\nu/kT} \)
Populations of states

- Populations of different energy levels depend on detailed processes that populate/depopulate them.
- In thermal equilibrium it’s easy – Boltzmann gives relative populations, otherwise hard.
- Population of a level with energy \( E_i \) above ground state and statistical weight \( g_i \) is:
  \[
  N_i = \frac{N}{U} g_i e^{-E_i/kT}
  \]

- \( N \) is the total number of atoms in all states per unit volume and \( U \) is the partition function:
  \[
  N = \sum_i N_i \implies U = \sum_i g_i e^{-E_i/kT}
  \]
\[ N = \sum_i N_i \implies U = \sum_i g_i e^{-E_i/kT} \]

• At low $T$ only the first term is significant so $U =$ statistical weight $g_1$ of ground state.

• Beware: At finite $T$, $g_i$ for higher states becomes large while Boltzmann factor $\exp(-E_i/kT)$ tends to a constant once $E_i$ approaches ionization energy.
  – Partition function sum diverges :-(
  – Idealized model of isolated atom breaks down due to loosely bound electrons interacting with neighbouring atoms.
  – Solution: cut off partition function sum at finite $n$, e.g. when Bohr orbit radius equals interatomic distance:
    \[ a_0 \approx 5 \times 10^{-11} Z^{-1} n^2 \quad m \approx N^{-1/3} \]
  – More realistic treatments must include plasma effects. In practice: don’t worry too much about how exactly to cut it off.
Masers (bound-bound)

• In thermal equilibrium, the excited states of an atom are less populated

\[
\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT} < 1 \quad \text{and}, \quad \frac{N_1}{g_1} > \frac{N_2}{g_2}
\]

• If some mechanism can put enough atoms into an upper state the normal population of the energy levels is turned into an inverted population,

\[
\frac{N_1}{g_1} < \frac{N_2}{g_2}
\]

• This leads to:

• A negative absorption coefficient – amplification!
• At microwave frequencies, astrophysical masers typically involve H$_2$O or OH
  – produce highly polarized radiation,
  – Have extremely high brightness temperatures (all radiation emitted in a narrow line).
Masers in NGC4258

Water vapour masers have been observed in the inner pc of the galaxy NGC4258

- Velocities trace Keplerian motion around a central mass.
- Strongest evidence for a black hole with mass $4 \times 10^7 \, M_\odot$.
- Measurement of proper motions provides geometric distance to the galaxy and estimate of the Hubble constant.
- Masers also seen in star forming regions.
- Herrnstein et al 1999, Nature 400, 539
Lecture 7 revision quiz

• Write down the equation balancing upward and downward radiative transition rates for a 2-level atom in a radiation field of mean intensity $J$

• Use Boltzmann’s law to fill in the step in the calculation between slide 1 and slide 2

• What do the Einstein coefficients $A_{21}$, $B_{21}$ and $B_{12}$ symbolise?

• What are their units?

• Why is there no $A_{12}$ coefficient?

• What is the use of the Partition function $U$?