1. Nebular radiative transfer problems involving scattering in the vicinity of a stellar source of photons often require us to compute the mean intensity $J_\nu$. Show that, for a star of radius $R$ whose photosphere emits isotropically with uniform specific intensity $I_\nu$, the mean intensity at distance $r$ from the star is

$$J_\nu = \frac{I_\nu}{2} \left( 1 - \sqrt{1 - \frac{R^2}{r^2}} \right).$$

Hence show that at the stellar surface $J_\nu = I_\nu / 2$ and that at large distances from the star, $J_\nu \rightarrow I_\nu / 4 r^2$.

2. Describe the physical picture of the stages in the expansion of an HII region into a uniform medium. Explain how recombinations in the interior of the region affect the ionizing flux at the front.

3. The O$^{++}$ ion has a triplet $^3P$ term for the ground state with J=0, 1, 2. There is a singlet $^1D_2$ term at energy $\Delta E$=kT above the ground state, and a singlet $^1S_0$ term at energy $\Delta E$=kT above the $^1D_2$ term. Downward transitions from $^1S_0$ to $^1D_2$ emit line photons with wavelength 4363 Å. Downward transitions from $^1D_2$ to $^3P_2$ and $^3P_1$ emit at 5007 Å and 4959 Å respectively. Sketch the energy-level diagram showing the five levels and the three lines. Given that the ratio of the lines' Einstein coefficients is $A(\lambda 5007) \sim 3 A(\lambda 4959)$, predict the observed flux ratio of these two lines in the low-density limit where spontaneous emission occurs faster than collisional de-excitation. Justify your reasoning.

4. A distant HII region in the Milky Way is found to have a Balmer recombination line flux ratio $H\alpha/H\beta = 4.0$. Given that this flux ratio is close to 2.86 for unreddened HII regions, and that the extinction $A_\lambda$ (in magnitudes) varies inversely with wavelength $\lambda$, calculate the de-reddening factors by which the observed H$\alpha$ and H$\beta$ line fluxes must be multiplied to remove the effects of extinction.

5. The electronic energy as a function of the internuclear separation $R$ in a diatomic molecule can be approximated using a potential,

$$E(R) = E_0 \left[ 1 - e^{-{(R-R_0)/L}} \right]^2 - E_0$$

where $E_0$, $R_0$, and $L$ are constants. Sketch this potential, and show that it has a minimum at $R = R_0$. 
6. In a Monte Carlo code that simulates emission and scattering in a spherical circumstellar shell, the radial dependence of the emissivity is $j(r) \propto (r / R)^{-\alpha}$, where $r$ is in the range $R < r < R_{\text{max}}$. Derive an expression for randomly sampling the radial location for emitting photons in the shell.

7. The Rayleigh scattering phase function is independent of azimuthal angle, $\phi$, and has the dependence on polar angle, $\theta$: $P(\theta) \propto 1 + \cos^2 \theta$. What is the normalization factor so that the scattering phase function is normalized over all solid angles? How would you choose $\theta$ and $\phi$ values to randomly choose a scattering direction? Hint: you may not be able to derive analytic expressions for randomly choosing both $\theta$ and $\phi$. 
1. Use definition of mean intensity and integrate over all solid angles occupied by the star:

\[ J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega \]

\[ = \frac{1}{4\pi} I_\nu \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin \theta_c d\theta_c \]

\[ = \frac{1}{2} I_\nu \left[ \cos \theta \right]_0^{\theta_c} \]

\[ = \frac{1}{2} I_\nu \left[ 1 - \sqrt{1 - \frac{R^2}{r^2}} \right] \]

In the asymptotic limit \( r \gg R \) the binomial expansion gives:

\[ \left( 1 - \frac{R^2}{r^2} \right)^{1/2} \approx \left( 1 - \frac{R^2}{2r^2} \right) \]

\[ \Rightarrow J_\nu \approx \frac{I_\nu}{4} \frac{R^2}{r^2} \]

2. See lecture notes!

3. See lecture notes on nebular density diagnostics for energy-level diagram of O++.

Given that both lines arise from the same upper state, they are populated at the same rate so the line intensity ratio is the ratio of the transition probabilities

\[ I_{5007}/I_{4959} = A_{5007}/A_{4959} = 3. \]

The line ratio \( I_{5007}/I_{4959} \) depends only on the A coefficients, because both lines originate from the same upper level and are depopulated only by spontaneous emission in the low-density limit.
4. The main clue to remember here is that the extinction $A_\lambda$ is proportional to the optical depth $\tau_\lambda$, and hence that $\tau_\lambda = \tau_{H\beta}(\lambda_{H\beta}/\lambda)$.

Hence

$$\frac{f_{H\alpha}}{f_{H\beta}} = \frac{f_{H\alpha,0} \exp(-\tau_{H\alpha})}{f_{H\beta,0} \exp(-\tau_{H\beta})}$$

$$\Rightarrow \log\left(\frac{f_{H\alpha}}{f_{H\beta}}\right) = \log\left(\frac{f_{H\alpha,0}}{f_{H\beta,0}}\right) - (\tau_{H\alpha} - \tau_{H\beta})$$

$$= \log\left(\frac{f_{H\alpha,0}}{f_{H\beta,0}}\right) - \tau_{H\beta}\left(\frac{\lambda_{H\beta}}{\lambda_{H\alpha}} - 1\right)$$

Rearrange and substitute observed flux ratios and known line wavelengths:

$$\tau_{H\beta} = \frac{\log(2.86) - \log(4.0)}{(4861/6563) - 1}$$

$$= 1.2986$$

$$\Rightarrow \tau_{H\alpha} = \tau_{H\beta} \times \frac{4861}{6563}$$

$$= 0.9618$$

Dereddening factors are thus

$$\exp(\tau_{H\beta}) = 3.664$$

$$\exp(\tau_{H\alpha}) = 2.616$$

[Sanity check: $2.86 \times 3.664 / 2.616 = 4.0$]
\[
\frac{d}{dR} E_0 \left[ 1 - e^{-\frac{(R-R_0)}{L}} \right]^2 - E_0 = 2E_0 \left( 1 - e^{-\frac{(R-R_0)}{L}} \right) \frac{d}{dR} \left( 1 - e^{-\frac{(R-R_0)}{L}} \right)
\]

= 0 when \( R = R_0 \),
so there is a minimum at \( R = R_0 \), where \( E(R) = E_0 \).

There is a faster and more intuitive way to deduce the value of \( R \) where the minimum occurs. The squared quantity inside the square brackets on the LHS cannot take values less than 0, and there is only one minimum. At \( R = R_0 \) this expression is equal to zero, so that must be the minimum.