Analytic Solutions I

- ERT: formal solution
- Operators
- Eddington-Barbier surface approximations
- Eddington-Barbier Limb Darkening

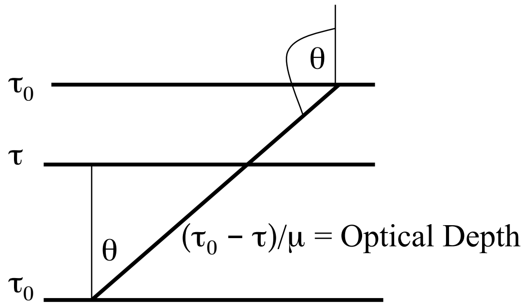
Formal Solution of ERT

$$\mu \frac{\mathrm{d}I_\nu(\tau_\nu,\mu)}{\mathrm{d}\tau_\nu} - I_\nu(\tau_\nu,\mu) = -S_\nu(\tau_\nu) \longrightarrow S_\nu \text{ isotropic}$$

Note here $d\tau = -\alpha$ ds
Multiply by integrating factor, $\exp(-t/\mu)$, with ν-dependence of optical depths implied, to get (Tutorial):

$$I_\nu(\tau,\mu) = \mathrm{e}^{-(\tau_0-\tau)/\mu} I_\nu(\tau_0,\mu) + \frac{1}{\mu}\int_\tau^{\tau_0} S_\nu(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt$$

Split into two regimes: Outward ($\mu > 0$), Inward ($\mu < 0$)



Split into two regimes: Outward ($\mu > 0$), Inward ($\mu < 0$)

Boundary conditions:

$\tau_0 \rightarrow$ infinity at lower boundary

No inward illumination: $I_\nu(\tau_0 = 0, \mu < 0) = 0$

$$\mu > 0: \quad I_\nu^+(\tau,\mu) = \int_\tau^\infty S_\nu(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt/\mu$$

$$\mu < 0: \quad I_\nu^-(\tau,\mu) = \int_0^\tau S_\nu(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt/|\mu|$$

Intensity measures source function weighted by $\exp(-t/\mu)$ along beam up to the point of interest

Moment Equations, Exponential Integrals, Operators

$$\int_{-1}^{1} I_\nu(\tau,\mu)\mu^n \,d\mu = \int_0^1 \mu^n \,d\mu \int_\tau^\infty S_\nu(t)\, e^{-(t-\tau)/\mu} \frac{dt}{\mu} + \int_{-1}^{0} \mu^n \,d\mu \int_0^\tau S_\nu(t)\, e^{-(\tau-t)/-\mu} \frac{dt}{-\mu}$$
$$= \int_\tau^\infty S_\nu(t)\, E_{n+1}(t-\tau)\,dt + (-1)^n \int_0^\tau S_\nu(t)\, E_{n+1}(t-\tau)\,dt$$

Exponential integrals E_n are defined by

$$E_n(x) \equiv \int_1^\infty \frac{e^{-xw}}{w^n}\,dw = \int_0^1 e^{-x/\mu} \mu^{n-1} \frac{d\mu}{\mu}$$

Tabulated in textbooks. For this course, we'll need approximations at small τ, so use:

$$E_n(0) = \frac{1}{n-1}$$

Schwarzschild-Milne Equations

Schwarzschild equation for the mean intensity:

$$J_\nu(\tau) \equiv \frac{1}{2}\int_{-1}^{1} I_\nu(\tau,\mu)\,d\mu$$
$$= \frac{1}{2}\int_\tau^\infty S_\nu(t)\,E_1(t-\tau)\,dt + \frac{1}{2}\int_0^\tau S_\nu(t)\,E_1(\tau-t)\,dt$$
$$= \frac{1}{2}\int_0^\infty S_\nu(t)\,E_1(|t-\tau|)\,dt$$

Milne equation for the flux:

$$\mathcal{F}_\nu(\tau_\nu) = \mathcal{F}_\nu^+(\tau_\nu) + \mathcal{F}_\nu^-(\tau_\nu)$$
$$= 2\pi\int_0^1 I_\nu(\tau_\nu)\,\mu\,d\mu - 2\pi\int_0^{-1} I_\nu(\tau_\nu)\,\mu\,d\mu$$
$$= 2\pi\int_{\tau_\nu}^\infty S_\nu(t_\nu)\,E_2(t_\nu-\tau_\nu)\,dt_\nu - 2\pi\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_2(\tau_\nu-t_\nu)\,dt_\nu$$

Completing the intensity moments in terms of exponential integrals, we get for the K_ν integral:

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)\, E_3(|\, t_\nu - \tau_\nu\, |)\, \mathrm{d}t_\nu$$

Surface Values

The emergent intensity and flux at the stellar surface are:

$$I_\nu^+(0,\mu) = \int_0^\infty S_\nu(\tau_\nu)\, \mathrm{e}^{-\tau/\mu}\, \mathrm{d}\tau_\nu / \mu$$

$$\mathcal{F}_\nu^+(0,\mu) = 2\pi \int_0^\infty S_\nu(\tau_\nu)\, E_2(\tau_\nu)\, \mathrm{d}\tau_\nu$$

Operators

Write the above equations in terms of *operators*. For the specific intensity, use the *Laplace Transform*:

$$\mathcal{L}_{1/\mu}\{S_\nu(\tau_\nu)\} \equiv \int_0^\infty S_\nu(t_\nu)\, \mathrm{e}^{-\tau/\mu}\, \mathrm{d}\tau_\nu / \mu = I_\nu^+(0,\mu)$$

In stellar atmospheres theory an important operator is the classical Lambda Operator, Λ_τ, defined by the RHS of the Schwarzschild eqn:

$$\Lambda_\tau\{f(\tau)\} \equiv \frac{1}{2}\int_0^\infty f(\tau)\, E_1(|\, t - \tau\, |)\, \mathrm{d}t$$

Operators

The Φ and χ operators are:

$$\Phi_\tau\{S_\nu(t_\nu)\} \equiv 2\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)\,E_2(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu - 2\int_0^{\tau_\nu} S_\nu(t_\nu)\,E_2(\tau_\nu-t_\nu)\,\mathrm{d}t_\nu$$
$$= F_\nu(\tau_\nu)$$

$$\chi_\tau\{S_\nu(t_\nu)\} \equiv 2\int_0^{\infty} S_\nu(t_\nu)\,E_3(t_\nu-\tau_\nu)\,\mathrm{d}t_\nu = 4K_\nu(\tau_\nu)$$

Some Properties

$$\Lambda_\tau\{1\} = 1 - \frac{1}{2}E_2(\tau)$$
$$\Lambda_\tau\{t\} = \tau - \frac{1}{2}E_3(\tau)$$
$$\Lambda_\tau\{t^2\} = \frac{2}{3} + \tau^2 - E_4(\tau)$$

Lambda operator applied to *S* gives *J*:

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^{\infty} S_\nu(t_\nu)\,E_1(|\,t-\tau_\nu\,|)\,\mathrm{d}t = \Lambda_{\tau_\nu}\{S_\nu(t_\nu)\}$$

Analytic Solutions

Optically thick radiation transfer only has analytic solutions at large depth: LTE holds, radiation field nearly isotropic In shallower, optically thinner layers, approximations are inevitable. Most important is the (first) *Eddington Approximation*, which we'll work up to.

Approximations based on power law expansions of $S(\tau)$

Recall, Taylor-McLaurin series:

$$f(\tau) = \sum_{n=0}^{\infty} a_n (\tau - \tau_0)^n$$

$$a_n = \left. \frac{f^n(\tau)}{n!} \right|_{\tau=\tau_0}$$

Surface: $\tau_0 = 0$
Interior: pick some τ_0

Approximations at the Surface

Eddington-Barbier Approximation:
This approximation for the emergent specific intensity is based on the polynomial expansion:

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} a_n \tau_\nu^n$$

which produces,:

$$I_\nu^+(0, \mu) = \mathcal{L}_{1/\mu}\{S_\nu(\tau_\nu)\} = \sum_{n=0}^{\infty} n!\, a_n\, \mu^n$$

(Tutorial Exercise)

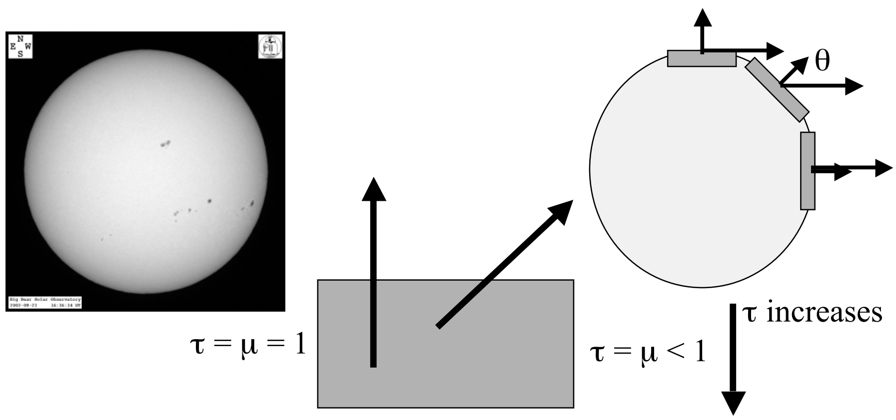
At surface, using operator forms for *J* and *F* gives (Tutorial):

$$
\begin{aligned}
I_\nu^+(0,\mu) &\approx a_0 + a_1\mu = S_\nu(\tau_\nu = \mu) \\
J_\nu(0) &\approx a_0 + \frac{2a_2}{3} - \frac{a_0}{2} + \frac{a_1}{4} - \frac{a_2}{3} \\
&\approx \frac{a_0}{2} + \frac{a_1}{4} + \frac{a_2}{3} \approx \frac{1}{2} S_\nu(\tau_\nu = 1/2) \\
F_\nu(0) &= a_0 + \frac{2a_1}{3} + a_2 + \ldots \approx S_\nu(\tau_\nu = 2/3)
\end{aligned}
$$

$$
\begin{aligned}
I_\nu^+(0,\mu) &\approx S_\nu(\tau_\nu = \mu) \\
J_\nu(0) &\approx \frac{1}{2} S_\nu(\tau_\nu = 1/2) \\
F_\nu(0) &\approx S_\nu(\tau_\nu = 2/3)
\end{aligned}
$$

These are the ***Eddington-Barbier Approximations*** for surface values of *I*, *J*, and *F* in the absence of external illumination (i.e., $I_\nu^-(0) = 0$).
They are exact for a source function that is linear with optical depth: $S_\nu(\tau_\nu) = a_0 + a_1\tau_\nu$ (Tutorial)

Limb Darkening



Towards limb, τ = μ is at a shallower depth.
If $S(\tau) = a + b\,\tau$ then…
$I = S(\tau = \mu) = a + b\,\mu$, so *I* (Limb) < *I* (Center)