Stellar mass limits

- Hydrostatic equilibrium is the key element.
- Combine ideal classical gas law: \( P_c = \frac{\rho_c}{m} kT_c \).
- with expression for central pressure to get:
  \[
  kT_c = \left( \frac{\pi}{36} \right)^{1/3} G\bar{m}M^{2/3} \rho_c^{1/3}.
  \]
- Applied to a contracting protostar, shows that \( T_c \) rises steadily as \( \rho_c \) increases.
- Temperature will continue to rise until either
  - Thermonuclear fusion starts to regulate \( T_c \); or
  - Electrons in core become degenerate
- Condition for stardom is that fusion sets in before electron degeneracy prevents further contraction.

Maximum core temperature

- Suppose star contracts until e\textsuperscript{}\textsuperscript{-} are degenerate in core but ions form a classical gas. Then:
  \[
  P_c = K_{NR} n_c^{5/3} + n_c kT_c
  \]
  \[
  = K_{NR} \left( \frac{\rho}{m_{\text{H}}} \right)^{5/3} + \frac{\rho}{m_{\text{H}}} kT_c \text{ for a pure H plasma.}
  \]
- Use hydrostatic equilibrium to eliminate \( P_c \):
  \[
  kT_c = \left( \frac{\pi}{36} \right)^{1/3} Gm_{\text{H}}M^{2/3} \rho_c^{1/3} - K_{NR} \left( \frac{\rho_c}{m_{\text{H}}} \right)^{2/3}.
  \]
- Has form
  \[
  kT_c = A\rho_c^{1/3} - B\rho_c^{2/3} \text{ which has maximum value}
  \]
  \[
  (kT_c)_{\text{max}} = A^2/4B \text{ at } \rho_{c,\text{max}} = (A/2B)^3.
  \]
Minimum mass of a star

- Substituting, get

\[(kT_c)_{\text{max}} = \left(\frac{\pi}{36}\right)^{2/3} \frac{G^2 m_{\text{H}}^{8/3}}{4K_{\text{NR}}} M^{4/3}.\]

- Need this max temp to be > \(T_{\text{ign}}\), so:

\[M_{\text{min}} = \left(\frac{36}{\pi}\right)^{1/2} \left(\frac{4K_{\text{NR}}}{G^2 m_{\text{H}}^{8/3}}\right)^{3/4} \left(kT_{\text{ign}}\right)^{3/4}.\]

For \(T_{\text{ign}} = 1.5 \times 10^6\) K, \(M_{\text{min}} = 0.1M_{\text{Sun}}\).

Radiation pressure in stellar core

\[P_c = P_{\text{gas}} + P_{\text{rad}},\] more conveniently written in terms of a parameter \(\beta:\)

\[P_{\text{gas}} = \beta P_c = \frac{P_c}{m} kT_c \quad \text{and} \quad P_{\text{rad}} = (1 - \beta)P_c = \frac{1}{3} aT_c^4.\]

Use \(\beta\) to eliminate \(T_c:\)

\[P_c = \left[\frac{3(1 - \beta)}{\alpha \beta^4}\right]^{1/3} \left[kP_c \frac{m}{\bar{m}}\right]^{4/3} \text{. Equate this to hydrostatic equilibrium pressure :}\]

\[\left[\frac{\pi}{36}\right]^{1/3} GM^{2/3} = \left[\frac{3(1 - \beta)}{\alpha \beta^4}\right]^{1/3} \left[kP_c \frac{m}{\bar{m}}\right]^{4/3}.\]
Maximum mass of a star

- Can use this equation to plot radiation pressure fraction \((1-\beta)\) as function of stellar mass.
- When \((1-\beta) > 0.5\) or so, radiation pressure destabilizes star -- so stars with \(M > 50 ~ M_{\text{Sun}}\) or so tend to blow apart

Understanding the main sequence

- Develop dimensionless equations of stellar structure for homologous models.
- Discover how mass-radius and mass-luminosity relations depend on opacity and energy generation.
- Determine slope of main sequence.
- Learn how changes in abundance affect location of main sequence in HR diagram.
Dimensionless structure equations

- Define \( x = \frac{r}{R} \), \( q = \frac{M(r)}{M} \), \( t = \frac{T}{T_0} \), \( p = \frac{P}{P_0} \).

- and use \( \rho = \frac{\mu m}{k} \frac{P}{T} \).

- Hydrostatic equilibrium becomes:

\[
\frac{p_0}{R} \frac{dp}{dx} = \frac{\mu m}{k} \frac{P_0}{T} \frac{p}{T_0} \frac{GM}{R^2} x^2 \implies \frac{dp}{dx} = \frac{p}{q} \frac{q}{x^2}
\]

where \( T_0 = \frac{\mu m GM}{kR} \).

Mass continuity + energy transport

- Similarly:

\[
\frac{M}{R} \frac{dq}{dx} = 4\pi R^2 x^2 \frac{\mu m}{k} \frac{P_0}{T} \frac{p}{T_0} \implies \frac{dq}{dx} = x^2 \frac{p}{t}
\]

where \( P_0 = \frac{GM^2}{4\pi R^4} \).

- Transport: Let \( \kappa = \kappa_0 \rho^\alpha T^{-\beta} = \kappa_0 \left( \frac{\mu m P_0}{k T_0 t} \right)^\alpha (T_0 t)^{-\beta} \).

\[
\frac{T_0}{R} \frac{dt}{dx} = -\frac{3\kappa_0}{16\pi a c (T_0 t)^3} \left( \frac{\mu m P_0}{k T_0 t} \right)^{1+\alpha} (T_0 t)^{-\beta} \frac{L f}{R^2 x^2}
\]

\[
\implies \frac{dt}{dx} = -C \frac{P^{1+\alpha}}{x^2} \frac{f}{t^{4+\alpha+\beta}}
\]

where \( C = \frac{3\kappa_0}{16\pi a c} \left( \frac{\mu m}{k} \right)^{1+\alpha} \frac{P_0^{1+\alpha}}{T_0^{5+\alpha+\beta}} \frac{L}{R} \).
Energy generation

- Energy generation:

\[ \varepsilon = \varepsilon_0 \rho T^\nu = \varepsilon_0 \frac{\mu m P_0 P}{k T_0} (T_0 t)^\nu. \]

\[ \frac{L}{R \, dx} = 4\pi r^2 \rho = 4\pi R^2 x^2 \varepsilon_0 \left( \frac{\mu m P_0 P}{k T_0} \right)^2 (T_0 t)^\nu \]

\[ \Rightarrow \frac{df}{dx} = D \frac{x^2 P^2}{t^{2-\nu}} \text{ where } D = \frac{4\pi \varepsilon_0 R^3}{L} \left( \frac{\mu m}{k} \right)^2 \frac{P_0^2}{T_0^{2-\nu}}. \]

Mass-radius relation: 1

- The constants \( P_0, T_0, C \) and \( D \) are unique to each star and apply throughout the entire structure.
- The product \( CD \) eliminates \( L \) to give a direct relation between mass and radius:

\[ CD = \frac{3\kappa_0\varepsilon_0 R^2}{4ac} \left( \frac{\mu m}{k} \right)^{3+\alpha} \frac{P_0^{3+\alpha}}{T_0^{7+\alpha+\beta-\nu}} \]

\[ = \frac{3\kappa_0\varepsilon_0 R^2}{4ac} \left( \frac{\mu m}{k} \right)^{3+\alpha} \left( \frac{GM^2}{4\pi R^4} \right)^{3+\alpha} \left( \frac{\mu m GM}{k R} \right)^{v-7-\alpha-\beta} \]

\[ = \frac{3\kappa_0\varepsilon_0}{4ac} \left( \frac{m}{k} \right)^{3+\alpha} \left( \frac{G}{4\pi} \right)^{3+\alpha} \left( \frac{mG}{k} \right)^{v-7-\alpha-\beta} \mu^{v-4-\beta} M^{v-1+\alpha-\beta} R^{v+3+3\alpha-\beta}. \]
Mass-radius relation: 2

- Hence get mass-radius relation:
  \[ R^{\nu+3\alpha-\beta} \propto M^{\nu-1+\alpha-\beta}. \]

- Kramers’ opacity law and CNO cycle give \( \alpha=1, \beta=7/2, \nu=16 \), so:
  \[ R \propto M^{12.5/18.5}. \]

- Böhm-Vitense finds that in practice, an opacity law with \( \alpha=0.5, \beta=2.5 \) gives a better fit:
  \[ R \propto M^{13/18}. \]