Stellar mass limits

- **Hydrostatic equilibrium is the key element.**
- **Combine ideal classical gas law:** $P_c = \frac{\rho_c}{\overline{m}} kT_c .$
- **with expression for central pressure to get:**

$$kT_c = \left(\frac{\pi}{36}\right)^{1/3} G\overline{m}M^{2/3}\rho_c^{\ 1/3} .$$

- **Applied to a contracting protostar, shows that T_c rises steadily as ρ_c increases.**
- **Temperature will continue to rise until either**
  - **Thermonuclear fusion starts to regulate T_c ; or**
  - **Electrons in core become degenerate**
- **Condition for stardom is that fusion sets in before electron degeneracy prevents further contraction.**

Maximum core temperature

- **Suppose star contracts until e^- are degenerate in core but ions form a classical gas. Then:**

$$P_c = K_{\mathrm{NR}} n_e^{5/3} + n_i kT_c$$

$$= K_{\mathrm{NR}} \left(\frac{\rho}{m_{\mathrm{H}}}\right)^{5/3} + \frac{\rho}{m_{\mathrm{H}}} kT_c \text{ for a pure H plasma.}$$

- **Use hydrostatic equilibrium to eliminate P_c:**

$$kT_c = \left(\frac{\pi}{36}\right)^{1/3} Gm_{\mathrm{H}}M^{2/3}\rho_c^{\ 1/3} - K_{\mathrm{NR}}\left(\frac{\rho_c}{m_{\mathrm{H}}}\right)^{2/3} .$$

- **Has form**

$$kT_c = A\rho_c^{\ 1/3} - B\rho_c^{\ 2/3} \text{ which has maximum value}$$

$$\left(kT_c\right)_{\max} = A^2/4B \text{ at } \rho_{c,\max} = (A/2B)^3 .$$

Minimum mass of a star

- **Substituting, get**

$$\left(kT_c\right)_{\max} = \left(\frac{\pi}{36}\right)^{2/3} \frac{G^2 m_H^{8/3}}{4K_{\mathrm{NR}}} M^{4/3}.$$

- **Need this max temp to be > T_{ign}, so:**

$$M_{\min} = \left(\frac{36}{\pi}\right)^{1/2} \left(\frac{4K_{\mathrm{NR}}}{G^2 m_H^{8/3}}\right)^{3/4} \left(kT_{\mathrm{ign}}\right)^{3/4}.$$

For $T_{\mathrm{ign}} = 1.5 \times 10^6$ K, $M_{\min} \approx 0.1 M_{\mathrm{Sun}}$.

Radiation pressure in stellar core

$P_{\mathrm{c}} = P_{\mathrm{gas}} + P_{\mathrm{rad}}$, more conveniently written in terms of a parameter β :

$$P_{\mathrm{gas}} = \beta P_{\mathrm{c}} = \frac{\rho_c}{\overline{m}} kT_c \text{ and } P_{\mathrm{rad}} = (1-\beta)P_{\mathrm{c}} = \frac{1}{3} aT_c^4.$$

Use β to eliminate T_c :

$$P_{\mathrm{c}} = \left[\frac{3}{a}\frac{(1-\beta)}{\beta^4}\right]^{1/3} \left[\frac{k\rho_c}{\overline{m}}\right]^{4/3}.$$ Equate this to hydrostatic equilibrium pressure :

$$\left[\frac{\pi}{36}\right]^{1/3} GM^{2/3} = \left[\frac{3}{a}\frac{(1-\beta)}{\beta^4}\right]^{1/3} \left[\frac{k\rho_c}{\overline{m}}\right]^{4/3}.$$

Maximum mass of a star

- **Can use this equation to plot radiation pressure fraction (1-β) as function of stellar mass.**
- **When (1-β) > 0.5 or so, radiation pressure destabilizes star -- so stars with M > 50 M_{Sun} or so tend to blow apart**

Understanding the main sequence

- **Develop dimensionless equations of stellar structure for homologous models.**
- **Discover how mass-radius and mass-luminosity relations depend on opacity and energy generation.**
- **Determine slope of main sequence.**
- **Learn how changes in abundance affect location of main sequence in HR diagram.**

Dimensionless structure equations

- **Define** $x = \frac{r}{R}, \quad q = \frac{M(r)}{M}, \quad t = \frac{T}{T_0}, \quad p = \frac{P_g}{P_0}.$
- **and use** $\rho = \frac{\mu m}{k}\frac{P_g}{T}.$
- **Hydrostatic equilibrium becomes:**

$$\frac{P_0}{R}\frac{dp}{dx} = \frac{\mu m}{k}\frac{P_0}{T_0}\frac{p}{t}\frac{GM}{R^2}\frac{q}{x^2} \Rightarrow \frac{dp}{dx} = \frac{p}{t}\frac{q}{x^2}$$

where $T_0 \equiv \frac{\mu m}{k}\frac{GM}{R}.$

Mass continuity + energy transport

- **Similarly:** $$\frac{M}{R}\frac{dq}{dx} = 4\pi R^2 x^2 \frac{\mu m}{k}\frac{P_0}{T_0}\frac{p}{t} \Rightarrow \frac{dq}{dx} = x^2 \frac{p}{t}$$

where $P_0 \equiv \frac{GM^2}{4\pi R^4}.$

- **Transport:** Let $\kappa = \kappa_0 \rho^\alpha T^{-\beta} = \kappa_0 \left(\frac{\mu m}{k}\frac{P_0 p}{T_0 t}\right)^\alpha (T_0 t)^{-\beta}.$

$$\frac{T_0}{R}\frac{dt}{dx} = -\frac{3\kappa_0}{16\pi ac(T_0 t)^3}\left(\frac{\mu m}{k}\frac{P_0 p}{T_0 t}\right)^{1+\alpha}(T_0 t)^{-\beta}\frac{Lf}{R^2 x^2}$$

$$\Rightarrow \frac{dt}{dx} = -C\frac{p^{1+\alpha}}{t^{4+\alpha+\beta}}\frac{f}{x^2} \text{ where } C = \frac{3\kappa_0}{16\pi ac}\left(\frac{\mu m}{k}\right)^{1+\alpha}\frac{P_0^{1+\alpha}}{T_0^{5+\alpha+\beta}}\frac{L}{R}.$$

Energy generation

- **Energy generation:**

$$\text{Let } \varepsilon = \varepsilon_0 \rho T^{\nu} = \varepsilon_0 \frac{\mu m}{k} \frac{P_0 p}{T_0 t} (T_0 t)^{\nu}.$$

$$\frac{L}{R}\frac{df}{dx} = 4\pi r^2 \varepsilon \rho = 4\pi R^2 x^2 \varepsilon_0 \left(\frac{\mu m}{k}\frac{P_0 p}{T_0 t}\right)^2 (T_0 t)^{\nu}$$

$$\Rightarrow \frac{df}{dx} = D\frac{x^2 p^2}{t^{2-\nu}} \text{ where } D = \frac{4\pi\varepsilon_0 R^3}{L}\left(\frac{\mu m}{k}\right)^2 \frac{P_0^2}{T_0^{2-\nu}}.$$

Mass-radius relation: 1

- **The constants P_0, T_0, C and D are unique to each star and apply throughout the entire structure.**
- **The product CD eliminates L to give a direct relation between mass and radius:**

$$CD = \frac{3\kappa_0 \varepsilon_0 R^2}{4ac}\left(\frac{\mu m}{k}\right)^{3+\alpha} \frac{P_0^{3+\alpha}}{T_0^{7+\alpha+\beta-\nu}}$$

$$= \frac{3\kappa_0 \varepsilon_0 R^2}{4ac}\left(\frac{\mu m}{k}\right)^{3+\alpha} \cdot \left(\frac{GM^2}{4\pi R^4}\right)^{3+\alpha} \left(\frac{\mu m}{k}\frac{GM}{R}\right)^{\nu-7-\alpha-\beta}$$

$$= \frac{3\kappa_0 \varepsilon_0}{4ac}\left(\frac{m}{k}\right)^{3+\alpha} \cdot \left(\frac{G}{4\pi}\right)^{3+\alpha} \left(\frac{mG}{k}\right)^{\nu-7-\alpha-\beta} \frac{\mu^{\nu-4-\beta} M^{\nu-1+\alpha-\beta}}{R^{\nu+3+3\alpha-\beta}}$$

Mass-radius relation: 2

- **Hence get mass-radius relation:**

$$R^{\nu+3+3\alpha-\beta} \propto M^{\nu-1+\alpha-\beta}.$$

- **Kramers' opacity law and CNO cycle give α=1, β =7/2, ν=16, so:**

$$R \propto M^{12.5/18.5}.$$

- **Böhm-Vitense finds that in practice, an opacity law with α=0.5, β=2.5 gives a better fit:**

$$R \propto M^{13/18}.$$