Mass-luminosity relation

- Use mass-radius relation and the expression for the constant $C$, again substituting for $P_0$ and $T_0$:

$$C = \frac{3k_0}{16\pi ac} \left( \frac{\mu m}{k} \right)^{1+\alpha} \left( \frac{GM^2}{4\pi R^3} \right)^{1+\alpha} \left( \frac{\mu m GM}{k R} \right)^{-5-\alpha-\beta} \frac{L}{R}$$

$$\Rightarrow L \propto \frac{R^{3\alpha-\beta}}{M^{\alpha-3-\beta}}.$$ 

- Use mass-radius relation to get $L$ as function of $M$ only, e.g. for $\alpha=0.5$, $\beta=2.5$ and $\nu=16$:

$$L \propto \frac{M^5}{R} \quad \text{and} \quad R \propto M^{13/18} \quad \Rightarrow \quad L \propto M^{5-13/18} \approx M^{4.3}.$$ 

Slope of main sequence

- Don’t confuse internal temperature with effective (surface) temperature!

- Since

$$T_{\text{eff}}^4 \propto \frac{L}{R^2},$$

- Get for $\alpha=0.5$ and $\beta=2.5$:

$$T_{\text{eff}}^4 \propto \frac{M^{5-13/18}}{M^{26/18}} \quad \Rightarrow \quad T_{\text{eff}}^4 \propto M^{0.71}$$

$$\Rightarrow \quad L \propto T_{\text{eff}}^{(5-13/18)/0.71} = T_{\text{eff}}^{6.04}$$

- Better fit to observed MS slope for early-type CNO burning stars than Kramers' Law ($\alpha=1$, $\beta=3.5$)
Heavy element abundance

- In deep interior, only heavy-element ions have enough electrons to contribute to b-f opacity.
- Hence $\kappa_0$ depends on heavy-element abundance.
- Note that $\epsilon_0$ also depends on heavy element ($^{14}$N) abundances if CNO is dominant energy source, but not if p-p dominates.
- Consider stars of one solar mass or less powered by p-p chain ($\nu=4$).
- Again eliminate L using product CD, including Z dependence of $\kappa_0$ and with $\alpha=0.5$, $\beta=2.5$:

$$CD \propto Z^{\frac{\mu}{\nu+4}-\beta} \frac{M^{\nu+4} \alpha-\beta}{R^{\nu+3+3 \alpha-\beta}} \propto Z \frac{M}{R^6}.$$  

Different metallicity, constant mass

- At fixed mass this gives only a very weak dependence of radius on metallicity: $R \propto Z^{1/6}$
- Insert into equation for C at constant M:

$$C \propto Z^{\frac{\mu}{\nu+4}-\beta} \frac{M^{\nu+4} \alpha-\beta}{R^{\nu+3+3 \alpha-\beta}} \propto ZM^5 LR \propto ZLR \propto Z^{7/6} L$$

$\Rightarrow L \propto Z^{-7/6}$.

and $T_{\text{eff}}^4 \propto \frac{L}{R^2} \propto \frac{1}{Z^{2/6} Z^{7/6}} \Rightarrow T_{\text{eff}} \propto Z^{-9/4}.$

- Hence get L-$T_{\text{eff}}$ relation:

$$Z \propto T_{\text{eff}}^{-24/9} \propto L^{-6/7} \Rightarrow L \propto T_{\text{eff}}^{28/9}.$$  

- Shallower than slope of main sequence.
Metallicity and the main sequence

• Main sequence moves blueward for low-metallicity systems.
• cf. location of globular-cluster main sequences in HR diagram.
• Also note that although metal poor “subdwarf” stars appear to lie below MS because they are less luminous at the same colour, they are really bluer and more luminous than metal-rich stars of the same mass!

Beyond the ZAMS

• So far we have taken no account of changes in composition as a star evolves.
• ZAMS star is initially well-mixed by convection.
• Subsequent H burning leads to
  – increase in molecular weight $\mu$ in core
  – decrease in core H abundance $X$.
• Again use CD with $\alpha=0.5$, $\beta=2.5$ at constant mass to get:
  $$ CD \propto Z \frac{\mu^{\gamma-4-\beta} M^{\gamma-1+\alpha-\beta}}{R^{\gamma+3+3\alpha-\beta}} \Rightarrow \mu^{\gamma-6.5} \propto R^{\gamma+2}. $$
• and get luminosity dependence from expression for $C$:
  $$ C \propto Z \frac{\mu^{\gamma-4-\beta} M^{\gamma-3+\alpha-\beta}}{R^{3\alpha-\beta}} L \Rightarrow L \propto \mu^{6.5} R^{-1}. $$
Why will the Sun become a red giant?

- Helium accumulates in core.
- Core shrinks and heats up to maintain hydrostatic equilibrium as mol. wt. increases:

\[ kT_c = \left( \frac{\pi}{36} \right)^{1/3} GmM^{2/3} \rho_c^{1/3}. \]

- Higher temperatures keep H burning going in thin shell surrounding He core, now dominated by CN cycle thanks to higher T.
- Luminosity and temperature are too great for low-mol. wt. material outside shell source.
- Excess internal pressure requires envelope to expand to maintain hydrostatic equilibrium.
- Shell source slowly moves out. Red giant expands further as He core mass grows.

What causes the helium flash?

- In stars with M < 2.25 \( M_{\text{Sun}} \) or so, helium core contracts until electrons become degenerate.
- Electron pressure now supports star, but temperature continues to increase.
- At core temp \( \sim 10^8 \) K, helium burning via \( 3\alpha \) reaction begins.
- Energy released by He burning raises T further but has no effect on pressure: thermonuclear thermostat doesn’t work.
- Thermal runaway raises temperature within seconds: brief but enormous increase in core luminosity “lifts” electron degeneracy.
- Star settles down in new equilibrium with expanded non-degenerate hot He-burning core.
White dwarfs

- Helium-burning phase ends with degenerate C-O core surrounded by layer of H, He.
- Electron degeneracy provides pressure support, so $\rho$ is independent of $T$.
- If electrons are NR, use previous arguments to get central density needed for support:

$$P = K_{NR} \left[ \frac{Y_e \rho_c}{m_H} \right]^{5/3} \approx \left[ \frac{\pi}{36} \right]^{1/3} GM^{2/3} \rho_c^{4/3}$$

$$\Rightarrow \rho_c \approx \frac{\pi}{36} \frac{G^2 M^2}{K_{NR}^3} \left[ \frac{m_H}{Y_e} \right]^{5} \text{ where } Y_e \equiv \frac{n_e m_H}{\rho_c}.$$  

Number of electrons per nucleon (1 for pure H, $\sim 0.5$ for anything heavier)

The Chandrasekhar limit

- Electrons at central density of a carbon WD with $M = 0.4M_{\odot}$ have $p_F \sim 0.65 m_e c$ and $\epsilon_F \sim 0.19 m_e c^2$.
- For more massive WD, approach equation of state for UR degenerate matter:

$$P = K_{UR} \left[ \frac{Y_e \rho_c}{m_H} \right]^{4/3} \approx \left[ \frac{\pi}{36} \right]^{1/3} GM^{2/3} \rho_c^{4/3}$$

- Central density cancels; get an expression for $M$ (in the limit of infinite central density) in terms of $Y_e$ and fundamental constants:

$$M_{CH} \approx \left[ \frac{K_{UR}}{G} \right]^{3/2} \left[ \frac{Y_e}{m_H} \right]^2 \left[ \frac{36}{\pi} \right]^{1/2} = 4.3Y_e^2 M_{\odot}.$$  

- More accurate polytropic treatment gives $M_{CH} \sim 1.4M_{\odot}$ for $Y_e \sim 0.5$ (2 electrons per nucleon).
Supernova collapse

- See AS2001 notes for details of advanced burning schemes leading to formation of $^{56}\text{Fe}$ core in stars with $M > 8M_{\text{Sun}}$.
- Contraction under gravity converts gravitational to internal energy:
  - If exothermic nuclear reactions result, internal KE and hence pressure increase, halting contraction BUT
  - If nuclear reactions result which ABSORB internal energy ("stellar refrigerator"), collapse proceeds unopposed.
- Nuclear photodisintegration:
  - $\gamma + ^{56}\text{Fe} \rightarrow ^{13}\text{He} + 4n$ absorbs $Q=(13m_4+4m_1-m_{56})c^2=124.4\text{ MeV}$
  - i.e. 1 kg of $^{56}\text{Fe}$ absorbs $2\times10^{14}\text{ J}$
- Electron capture via inverse beta decay:
  - $e^- + ^{56}\text{Fe} \rightarrow ^{56}\text{Mn} + \nu_e$ -- neutrino escapes carrying energy.
  - threshold energy = $m_e c^2+3.7\text{ MeV}$.

Neutrino burst at core collapse

- Fermi energy of electrons in degenerate core equals $E_{\text{thresh}}$ when $\rho_e=1.1\times10^{12}\text{ kg m}^{-3}$.
- Can also have $e^- + ^{56}\text{Mn} \rightarrow ^{56}\text{Cr} + \nu_e$
- At densities $>2\times10^{13}\text{ kg m}^{-3}$, electron capture on to $^{56}\text{Cr}$ begins and proceeds rapidly.
- Average electron energy $\sim 10\text{ MeV}$ at this point.
- Iron core with $M=M_{\text{CH}}$ contains $\sim10^{57}$ electrons which can produce $10^{57}$ neutrinos, carrying away $1.6\times10^{45}\text{ J}$ on free-fall timescale:
  $$t_{\text{ff}} = \left(\frac{3\pi}{32G\rho}\right)^{1/2} \sim 10^{-3}\text{ s}.$$  
- Neutrino mean free path $\sim$ stellar radius (few km) at $\rho\sim10^{14}\text{ kg m}^{-3}$ -- trapped for a few seconds before diffusing out of core.
THE END