2.4 Energy conservation II: Transport.

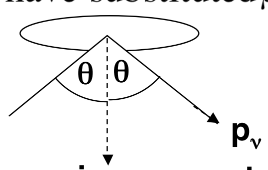
- **The Sun's interior is hotter than its surface.**
- **Existence of a temperature gradient implies an outward flux of energy.**
- **Energy flux is determined by conservation of energy as just shown.**
- **Temperature gradient depends on method of energy transport:**
  - **Radiative diffusion**
  - **Convective motions**

Flux and radiation pressure

- **From Nebulae/Atmospheres, flux through surface element dA in frequency interval from n to ν + dν is:**

$$F_\nu = \oint_{4\pi} I_\nu \cos\theta . d\Omega = 2\pi \int_{-1}^{1} I_\nu \mu d\mu$$

where we have substituted $\mu = \cos\theta$.



- **Each photon carries momentum p_ν = E/c = hν/c**
- **Bounces off dA at angle of incidence cos θ .**
- **Momentum transferred per photon= 2 $p_\nu$cos θ .**

Pressure = momentum flux

- **Pressure = outward (photons/sec/unit area) x (2 $p_\nu$cos θ)**
  = (in+out)(photons/sec/unit area) x ($p_\nu$cos θ)

$$P_{\mathrm{rad},\nu} = \frac{1}{c} \oint_{4\pi} I_\nu \cos^2\theta . d\Omega = \frac{2\pi}{c} \int_{-1}^{1} I_\nu \mu^2 d\mu$$

- **The factor cos^2 θ allows for both foreshortening of the surface element's cross-section and transfer of normal component of momentum.**

Radiative energy transport

- **Specific intensity of beam travelling at angle θ to radial direction in medium of density ρ, opacity κ_ν and source function S_ν:**

$$\cos\theta \frac{dI_\nu}{dr} = \rho\kappa_\nu (S_\nu - I_\nu). \text{ Multiply both sides by}$$

$$\mu = \cos\theta, \text{ and integrate:}$$

$$\int_{-1}^{1} \mu^2 \frac{dI_\nu}{dr} d\mu = \rho\kappa_\nu \int_{-1}^{1} \mu \, d\mu (S_\nu - I_\nu)$$

S_ν is isotropic

$$\Rightarrow \frac{d}{dr} \int_{-1}^{1} I_\nu \mu^2 d\mu = \rho\kappa_\nu S_\nu \underbrace{\cancel{\int_{-1}^{1} \mu \, d\mu}}_{0} - \rho\kappa_\nu \int_{-1}^{1} I_\nu \mu \, d\mu$$

cf.Radiation pressure **cf. Flux**

Radiative-equilibrium temperature gradient

- **We find that the opacity determines the temperature gradient:**

$$c\frac{dP_{\mathrm{rad},\nu}}{dr} = -\kappa_\nu \rho F_\nu \Rightarrow \frac{dP_{\mathrm{rad}}}{dr} = -\frac{\kappa\rho}{c}F$$

where we define a flux - weighted mean opacity,

$$\kappa \equiv \int_0^\infty \kappa_\nu F_\nu d\nu \Bigg/ \int_0^\infty F_\nu d\nu \text{. But } P_{\mathrm{rad}} = \frac{u}{3} = \frac{1}{3}aT^4,$$

$$\text{so } \frac{dP_{\mathrm{rad}}}{dr} = \frac{4}{3}aT^3\frac{dT}{dr}. \quad \text{Also} \quad F = \frac{L}{4\pi r^2},$$

$$\Rightarrow \frac{dT(r)}{dr} = -\frac{3\kappa\rho}{4acT^3}\frac{L(r)}{4\pi r^2}$$

Another equation and a new variable, T(r).

Convective equilibrium

- **Suppose temperature gradient is radiative.**
- **Is it stable to small local perturbations?**
- **Suppose a blob of mass δm at radius r has its temperature perturbed by a small amount:**

$$\Delta T(r) = T_{\delta m}(r) - T(r).$$

- **Pressure will change by**

$$\Delta P(r) = P_{\delta m}(r) - P(r).$$

- **but pressure balance is quickly restored by a change in volume, to give density difference from surroundings:**

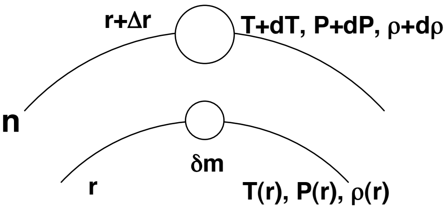
$$\Delta\rho(r) = \rho_{\delta m}(r) - \rho(r).$$

- **Temperature excess with pressure equilibrium in ideal gas → density deficiency → buoyancy.**

Buoyant stability

- **If $\Delta T > 0$, buoyant force > gravity, so blob rises to new position at $r+\Delta r$.**
- **Surroundings at new position have density**

$$\rho(r) + \frac{d\rho}{dr}\Delta r$$



- **while blob changes density to match local pressure:**

$$\rho_{\delta m}(r) + \left(\frac{d\rho}{dr}\right)_{\delta m}\Delta r$$

- **Element is stable if it becomes denser than surroundings, i.e. if:**

$$\left|\left(\frac{d\rho}{dr}\right)\right| > \left|\left(\frac{d\rho}{dr}\right)_{\delta m}\right|$$

- **Ideal gas:**

$$P \propto \rho T \Rightarrow \left|\left(\frac{dT}{dr}\right)\right| < \left|\left(\frac{dT}{dr}\right)_{\delta m}\right| \text{ for stability.}$$

Adiabatic changes

- **Rising blob is hotter than its surroundings as it rises so can only lose heat (& vice versa for falling blob).**
- **Hence change in temperature with r must be less than adiabatic (no heat loss) value :**

$$\left|\left(\frac{dT}{dr}\right)_{\delta m}\right| < \left|\left(\frac{dT}{dr}\right)_{\text{ad}}\right|.$$

- **Adiabatic gradient is given by:**

$$PV^{\gamma} = \text{const} \Rightarrow P \propto \rho^{\gamma} \propto \left(\frac{P}{T}\right)^{\gamma}, \text{ where } \gamma = \frac{C_P}{C_V}$$

$$\Rightarrow P^{1-\gamma}T^{\gamma} = \text{const}$$

$$\Rightarrow (1-\gamma)\frac{dP}{dr} + \gamma\frac{P}{T}\frac{dT}{dr} = 0.$$

Logarithmic T-P gradients

- **For an adiabatic blob, we thus get:**

$$\frac{1}{T}\left(\frac{dT}{dr}\right)_{\mathrm{ad}} = \frac{\gamma - 1}{\gamma}\frac{1}{P}\left(\frac{dP}{dr}\right)$$

- **or:**

$$\left(\frac{d\log T}{d\log P}\right)_{\mathrm{ad}} = \frac{\gamma - 1}{\gamma}.$$

- **Use hydrostatic equilibrium to get a similar T-P relation for the radiative gradient:**

$$\left(\frac{d\log T}{d\log P}\right)_{\mathrm{rad}} = \frac{3\kappa L(r)P}{16\pi\, acT^4 GM}.$$

Convective stability criterion

- **Remembering that pressure is the same inside and outside blob at all times, we can write the stability criterion as:**

$$\left|\left(\frac{d\log T}{d\log P}\right)_{\mathrm{rad}}\right| < \left|\left(\frac{d\log T}{d\log P}\right)_{\mathrm{ad}}\right|$$

$$\Rightarrow \frac{3\kappa L(r)P}{16\pi\, acT^4 GM} < \frac{\gamma - 1}{\gamma}.$$

Convectively unstable regions

- **(1) Cores of massive stars:**

  Radiation flux $L(r)/4\pi r^2$ can become very large while opacity $\kappa\rho$ remains small in the centres of main massive main - sequence stars.

- **(2) Outer envelopes of cool stars:**

  Adiabatic exponent γ can approach unity in sub - surface ionization zones in cool stars. Hence $(\gamma - 1)/\gamma$ can become small, *a*nd convection will set in at quite low values of $\left|(dT/dr)_{\mathrm{rad}}\right|$.

Energy transport

- **In formulating stellar structure problem, use a single expression for the temperature gradient:**

$$\frac{d\log T}{d\log P} = (1-\xi)\left(\frac{d\log T}{d\log P}\right)_{\mathrm{rad}} + \xi\left(\frac{d\log T}{d\log P}\right)_{\mathrm{ad}}.$$

where ξ characterizes the convective efficiency :

$\xi = 0 \Rightarrow$ radiative equilibrium

$\xi = 1 \Rightarrow$ adiabatic convection

$0 < \xi < 1 \Rightarrow$ non – adiabatic convection : ξ must be determined from convection theory.

2.5.1 Constitutive relations

- **Still need additional equations to describe ρ, ε, κ, ξ and (dlogT/dlogP) in terms of:**
  - **the state variables T and P, and**
  - **the composition of the stellar material (X,Y,Z or X_i)**
- **The following *constitutive relations* close the system of ODEs:**

$$\rho = \rho(P,T,X_i) \qquad \text{(equation of state)}$$

$$\varepsilon = \varepsilon(\rho,T,X_i) \qquad \text{(nuclear energy generation rate)}$$

$$\kappa = \kappa(\rho,T,X_i) \qquad \text{(opacity)}$$

$$\xi = \xi(\rho,T,X_i) \qquad \text{(convective efficiency)}$$

$$\frac{\mathrm{dlogT}}{\mathrm{dlogP}}(\rho,T,\xi,\kappa,X_i) \qquad \text{(energy transport)}$$

2.5.2 Equations of stellar structure

- **We have now determined the four basic (time-independent) equations of stellar structure.**
- **Use mass continuity to transform them to have enclosed mass as the independent variable.**
- **Mass continuity:** $$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$
- **Conservation of energy:** $$\frac{dL}{dm} = \varepsilon$$
- **Hydrostatic equilibrium:** $$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$
- **Energy transport:** $$\frac{dT}{dm} = \frac{Gm}{4\pi r^4}\frac{T}{P}\frac{d\log T}{d\log P}$$

2.5.3 Boundary conditions

- **To solve a system of n ODEs, we need to specify n boundary conditions.**
- **In Lagrangian frame, boundaries are at the centre (m=0) and the surface (m=M).**

At the centre:

$$r(m=0)=0$$

$$L(m=0)=0$$

At the surface :

$$T(m=M)=T_{\text{eff}}\left(=\frac{L}{4\pi R^2\sigma}\right)^{1/4}$$

$$P_{\text{gas}}(m=M)=0$$

2.5.4 Solution

- **Solution of equations of stellar structure gives the run of P, T, m and L as functions of r throughout the domain 0 < r < R.**
- **Solutions are characterized uniquely by**
  - **Total mass of star M = m(R)**
  - **Run of chemical composition through star.**
  - **Gravitational binding energy.**
- **Gives quantitative description of stellar interior.**