3.3.5 Radiative stars

- **Repeat for mass continuity, energy generation and radiative energy transport to get 4 simultaneous eqs in the four indices α...**

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & \chi_\rho & 0 & \chi_T \\ 0 & \lambda & -1 & \nu \\ 4 & -n & -1 & 4+s \end{pmatrix} \begin{pmatrix} \alpha_R \\ \alpha_\rho \\ \alpha_L \\ \alpha_T \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

- **Determinant of matrix in LHS is:**

$$D_{\text{rad}} = (3\chi_\rho - 4)(\nu - s - 4) - \chi_T(3\lambda + 3n + 4).$$

Denotes diffusive radiative transfer in energy transport equation; $D_{rad} \neq 0$.

3.3.6 Solutions

- **Solutions of matrix equation are:**

$$\alpha_R = \frac{1}{3D_{\text{rad}}}\left[1 - 2\left(\chi_T + \nu - s - 4\right)\right]$$

$$\alpha_\rho = \frac{2}{D_{\text{rad}}}\left[\chi_T + \nu - s - 4\right]$$

$$\alpha_T = 1 + \frac{1}{D_{\text{rad}}}\left[2\lambda\left(\chi_T + \nu - s - 4\right) - 2\nu(\chi_\rho + \lambda + n)\right]$$

$$\alpha_L = -\frac{2}{D_{\text{rad}}}\left[\chi_\rho + \lambda + n\right]$$

3.3.7 Convective stars

- **Energy transport by fully adiabatic convection gives:**

$$T \propto \rho^{\gamma-1} \qquad \text{where} \quad \gamma - 1 \equiv \left(\frac{d\ln T}{d\ln\rho}\right)_{\text{ad}} \text{ is the}$$

adiabatic thermodynamic derivative.

- **Matrix equation becomes:**

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & \chi_\rho & 0 & \chi_T \\ 0 & \lambda & -1 & \nu \\ 0 & \gamma - 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_R \\ \alpha_\rho \\ \alpha_L \\ \alpha_T \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

- **Slightly simpler determinant and solutions.**

3.3.8 Upper-main sequence stars

- **Dominant opacity source is electron scattering which is independent of ρ, T so n = s = 0.**
- **Energy source is thermonuclear CNO cycle for which $\lambda = 1$ and $\nu \sim 15$.**
- **Equation of state: ideal gas ($P \propto \rho T$) so $\chi_\rho = \chi_T = 1$.**
- **Substitute these values in equations for radiative stars to get:**

$$\alpha_R = 0.78, \alpha_L = 3.0, \alpha_\rho = -1.33, \alpha_T = 0.22.$$

Upper MS stars

- **Hence for M > 1 M_{Sun}:**

$$\frac{R}{R_{\mathrm{Sun}}} = \left(\frac{M}{M_{\mathrm{Sun}}}\right)^{0.78}, \frac{L}{L_{\mathrm{Sun}}} = \left(\frac{M}{M_{\mathrm{Sun}}}\right)^{3.0},$$

$$\frac{\rho}{\rho_{\mathrm{Sun}}} = \left(\frac{M}{M_{\mathrm{Sun}}}\right)^{-1.33}, \frac{T}{T_{\mathrm{Sun}}} = \left(\frac{M}{M_{\mathrm{Sun}}}\right)^{0.22}.$$

- **Not too bad: recall that $L \propto M^{3.6}$ for high-mass stars in binaries.**
- **Average density falls, temperature rises with increasing mass.**
- **But we've made some major approximations and haven't solved any DEs!**

Interlude: The Virial Theorem

- **Whole system is in equilibrium if hydrostatic equilibrium is satisfied at all r.**
- **Hence get relation between average internal pressure and gravitational PE of whole system.**

Multiply both sides of H.E. by $4\pi r^3$ and integrate from $r = 0$ to $r = R$ to get:

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = -\int_0^R \frac{Gm}{r} 4\pi r^2 \rho \, dr.$$

Average internal pressure

Integrate lhs by parts ($du = dP/dr, dv = 4\pi r^3$)

and substitute $dm = 4\pi r^2 \rho\, dr$:

$$\left[4\pi r^3 P\right]_0^R - 3\int_0^R 4\pi r^2 P\, dr = -\int_0^M \frac{Gm}{r}\, dm.$$

$P(R) = 0 \Rightarrow$ first term is zero. Subst. $dv = 4\pi r^2\, dr$:

$$-3\int_0^V P\, dv \left(\equiv 3\langle P\rangle V\right) = -\int_0^M \frac{Gm}{r}\, dm \left(\equiv E_{\mathrm{grav}}\right).$$

Volume-averaged pressure

Volume of system

Gravitational PE of system

Pressure: nonrelativistic gas

$$P = nkT = \frac{kT}{V} \quad \text{and } E_{\mathrm{kin}} = \frac{3}{2}kT$$

$$\Rightarrow P = \frac{2}{3}\frac{E_{\mathrm{kin}}}{V}.$$

So the gravitational and kinetic energies are related by:

$$2E_{\mathrm{kin}} + E_{\mathrm{grav}} = 0$$

and the total energy of the system is

$$E_{\mathrm{tot}} = -E_{\mathrm{kin}} = \frac{1}{2}E_{\mathrm{grav}}.$$

Fundamentally important result! Means that tightly bound systems in hydrostatic equilibrium have high particle KE, i.e. they're HOT.

Pressure: ultra-relativistic gas

- **For a gas of ultra-relativistic particles:**

$$E_{\rm kin} = 3kT \Rightarrow P = \frac{1}{3}\frac{E_{\rm kin}}{V}.$$

So the gravitational and kinetic energies are related by:

$$E_{\rm kin} + E_{\rm grav} = 0,$$

i.e. hydrostatic equilibrium is possible only if the total energy is zero.

- **As UR limit is approached, binding energy decreases and system is easily disrupted.**

Common ultra-relativistic gases

- ***Radiation-pressure dominated:***
- **Pressure from photons, UR by definition!**
  - **e.g. supermassive stars.**
- ***Degenerate electron-pressure dominated:***
- **If electron temperature becomes very high**
  - **e.g. massive white dwarfs.**
- **Both to be discussed in more detail later.**