3.3.5 Radiative stars

- Repeat for mass continuity, energy generation and radiative energy transport to get 4 simultaneous eqs in the four indices $\alpha$...

$$
\begin{pmatrix}
3 & 1 & 0 & 0 \\
4 & \chi_p & 0 & \chi_T \\
0 & \lambda & -1 & \nu \\
4 & -n & -1 & 4 + s
\end{pmatrix}
\begin{pmatrix}
\alpha_R \\
\alpha_p \\
\alpha_L \\
\alpha_T
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2 \\
-1 \\
1
\end{pmatrix}
$$

- Determinant of matrix in LHS is:

$$
D_{rad} = (3\chi_p - 4)(\nu - s - 4) - \chi_T(3\lambda + 3n + 4).
$$

Denotes diffusive radiative transfer in energy transport equation; $D_{rad} \neq 0$.

3.3.6 Solutions

- Solutions of matrix equation are:

$$
\alpha_R = \frac{1}{3D_{rad}} [1 - 2(\chi_T + \nu - s - 4)]
$$

$$
\alpha_p = \frac{2}{D_{rad}} [\chi_T + \nu - s - 4]
$$

$$
\alpha_T = 1 + \frac{1}{D_{rad}} [2\lambda(\chi_T + \nu - s - 4) - 2\nu(\chi_p + \lambda + n)]
$$

$$
\alpha_L = -\frac{2}{D_{rad}} [\chi_p + \lambda + n]
$$
3.3.7 Convective stars

- Energy transport by fully adiabatic convection gives:
  \[ T \propto \rho^{\gamma^{-1}} \]
  where \( \gamma - 1 = \left( \frac{d \ln T}{d \ln \rho} \right)_{\text{ad}} \) is the adiabatic thermodynamic derivative.

- Matrix equation becomes:
  \[
  \begin{pmatrix}
  3 & 1 & 0 & 0 \\
  4 & \chi_\rho & 0 & \chi_T \\
  0 & \lambda & -1 & \nu \\
  0 & \gamma - 1 & 0 & -1
  \end{pmatrix}
  \begin{pmatrix}
  \alpha_R \\
  \alpha_\rho \\
  \alpha_L \\
  \alpha_T
  \end{pmatrix}
  =
  \begin{pmatrix}
  1 \\
  2 \\
  -1 \\
  0
  \end{pmatrix}
  \]
  Slightly simpler determinant and solutions.

3.3.8 Upper-main sequence stars

- Dominant opacity source is electron scattering which is independent of \( \rho, T \) so \( n = s = 0 \).
- Energy source is thermonuclear CNO cycle for which \( \lambda = 1 \) and \( \nu \sim 15 \).
- Equation of state: ideal gas (\( P \propto \rho T \)) so \( \chi_\rho = \chi_T = 1 \).
- Substitute these values in equations for radiative stars to get:
  \[\alpha_R = 0.78, \alpha_L = 3.0, \alpha_\rho = -1.33, \alpha_T = 0.22.\]
Upper MS stars

- Hence for $M > 1 \ M_{\text{Sun}}$:
  \[
  \frac{R}{R_{\text{Sun}}} = \left( \frac{M}{M_{\text{Sun}}} \right)^{0.78}, \quad \frac{L}{L_{\text{Sun}}} = \left( \frac{M}{M_{\text{Sun}}} \right)^{3.0},
  \]
  \[
  \frac{\rho}{\rho_{\text{Sun}}} = \left( \frac{M}{M_{\text{Sun}}} \right)^{-1.33}, \quad \frac{T}{T_{\text{Sun}}} = \left( \frac{M}{M_{\text{Sun}}} \right)^{0.22}.
  \]
- Not too bad: recall that $L \propto M^{3.8}$ for high-mass stars in binaries.
- Average density falls, temperature rises with increasing mass.
- But we’ve made some major approximations and haven’t solved any DEs!

Interlude: The Virial Theorem

- Whole system is in equilibrium if hydrostatic equilibrium is satisfied at all $r$.
- Hence get relation between average internal pressure and gravitational PE of whole system.

Multiply both sides of H.E. by $4\pi r^3$ and integrate from $r = 0$ to $r = R$ to get:

\[
\int_{0}^{R} 4\pi r^3 \frac{dP}{dr} \ dr = -\int_{0}^{R} Gm \ 4\pi r^2 \rho \ dr.
\]
Average internal pressure

Integrate lhs by parts \((du = dP/dr, dv = 4\pi r^3)\) and substitute \(dm = 4\pi r^2 \rho \, dr\):

\[
\left[ 4\pi r^3 P \right]_0^R - 3\int_0^R 4\pi r^2 P \, dr = -\int_0^M \frac{Gm}{r} \, dm.
\]

\(P(R) = 0 \Rightarrow\) first term is zero. Subst. \(dv = 4\pi r^2 \, dr\):

\[-3\int_0^V P \, dv \left( \equiv \frac{3}{V} PV \right) = -\int_0^M \frac{Gm}{r} \, dm \left( \equiv E_{\text{grav}} \right).\]

Fundamentally important result!

Means that tightly bound systems in hydrostatic equilibrium have high particle KE, i.e. they’re HOT.

Pressure: nonrelativistic gas

\[P = n k T = \frac{k T}{V} \quad \text{and} \quad E_{\text{kin}} = \frac{3}{2} k T\]

\[\Rightarrow P = \frac{2}{3} \frac{E_{\text{kin}}}{V}.
\]

So the gravitational and kinetic energies are related by:

\[2E_{\text{kin}} + E_{\text{grav}} = 0\]

and the total energy of the system is

\[E_{\text{tot}} = -E_{\text{kin}} = \frac{1}{2} E_{\text{grav}}.\]
Pressure: ultra-relativistic gas

- For a gas of ultra-relativistic particles:

\[ E_{\text{kin}} = 3kT \Rightarrow P = \frac{1}{3} \frac{E_{\text{kin}}}{V}. \]

So the gravitational and kinetic energies are related by:

\[ E_{\text{kin}} + E_{\text{grav}} = 0, \]

i.e. hydrostatic equilibrium is possible only if the total energy is zero.

- As UR limit is approached, binding energy decreases and system is easily disrupted.

Common ultra-relativistic gases

- *Radiation-pressure dominated:*
  - Pressure from photons, UR by definition!
    - e.g. supermassive stars.

- *Degenerate electron-pressure dominated:*
  - If electron temperature becomes very high
    - e.g. massive white dwarfs.

- Both to be discussed in more detail later.