AS4023 Stars, Tutorial 1

1. Compute the ratio of ionized to neutral hydrogen in a container of pure hydrogen gas with temperature $T = 10^4$ K and electron pressure $P_e = 30$ Pa. The ionization potential of neutral hydrogen is 13.6 eV, and the partition functions of neutral and ionized hydrogen have values 2 and 1 respectively. Note: in exam conditions you will not be expected to derive, but will be expected to know the convenient logarithmic forms of the Saha and Boltzmann equations.

2. Starting from the differential contributions to the specific intensity by the emission and extinctions coefficients over a pathlength ds, derive the equation of radiation transfer along a ray in the form

$$\frac{dI}{d\tau} = S - I,$$

defining the optical depth τ, and source function S.

3. The transfer equation for radiation emerging from a stellar atmosphere at angle θ to the outward normal is,

$$\mu \frac{dI}{d\tau} = I - S\,,$$

where $\mu = \cos\theta$ and $d\tau = -\rho\kappa\, dr$. Derive the formal solution for the equation of radiation transfer between optical depths τ_0 and τ in the form

$$I(\tau,\mu) = \mathrm{e}^{-(\tau_0-\tau)/\mu} I(\tau_0,\mu) + \frac{1}{\mu}\int_\tau^{\tau_0} S(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt\,.$$

What are the upper and lower boundary conditions for a plane-parallel model atmosphere? Impose these boundary conditions and show that the specific intensity in the outward and inward directions is

$$\mu > 0: \quad I^+(\tau,\mu) = \int_\tau^\infty S(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt/\mu$$

$$\mu < 0: \quad I^-(\tau,\mu) = \int_0^\tau S(t)\ \mathrm{e}^{-(t-\tau)/\mu}\ dt/|\mu|$$

If the exponential integrals $E_n(x)$ are defined by

$$E_n(x) \equiv \int_1^\infty \frac{\mathrm{e}^{-xw}}{w^n}\,\mathrm{d}w = \int_0^1 \mathrm{e}^{-x/\mu}\mu^{n-1}\,\frac{\mathrm{d}\mu}{\mu},$$

show that the mean intensity and the flux are given by

$$J(\tau) = \frac{1}{2}\int_0^\infty S(t)\,E_1(|t-\tau|)\,\mathrm{d}t$$

$$\mathcal{F}(\tau) = 2\pi\int_\tau^\infty S(t)\,E_2(t-\tau)\,\mathrm{d}t - 2\pi\int_0^\tau S(t)\,E_2(\tau-t)\,\mathrm{d}t$$

4. If the source function in the Solar photosphere has a linear dependence upon the optical depth τ,

$$S = a_0 + a_1\tau \ ,$$

then show that the specific intensity of light emerging from the photosphere (i.e., I^+ in question 3) has a linear dependence upon cos θ.

If, more generally,

$$S = \sum_{i=0}^{i=n} a_i\tau^i \ ,$$

then show that,

$$I(\theta) = \sum_{i=0}^{i=n} A_i \cos^i\theta \ ,$$

and derive an expression for the A_i. In solving the above, you may use

$$\int_0^\infty x^i e^{-x} dx = i!$$

5. Using the general power series expansion for the source function given above, the properties of the Lambda operator given in the notes, and the equations for the specific intensity, mean intensity and flux, show that the surface values ($\tau = 0$) are given by

$$I^+(0,\mu) \approx a_0 + a_1\mu = S(\tau = \mu)$$

$$J(0) \approx a_0 + \frac{2a_2}{3} - \frac{a_0}{2} + \frac{a_1}{4} - \frac{a_2}{3}$$

$$\approx \frac{a_0}{2} + \frac{a_1}{4} + \frac{a_2}{3} \approx \frac{1}{2}S(\tau = 1/2)$$

$$F(0) = a_0 + \frac{2a_1}{3} + a_2 + \ldots \approx S(\tau = 2/3)$$

6. The equation of hydrostatic equilibrium is

$$\frac{dP}{dz} = -\rho\, g \ .$$

Re-write this in the form used to model a stellar atmosphere relating the pressure, optical depth, opacity, and gravity

If κ is directly proportional to the gas pressure P_g, show how P_g and κ would vary with optical depth τ. For a given optical depth show how the gas pressure varies with gravity. If the temperature distribution is,

$$T^4 = \frac{3}{4}T_e^4(\tau + 2/3)$$

then show that the temperature will be constant at small optical depths, but that in deeper layers $T \propto \sqrt{P_g}$.

7. At fixed pressure P_e = 10 Pa, calculate the fraction of hydrogen that is H^- as a function of temperature T. Likewise, calculate the fraction of hydrogen in the n = 3 level that contributes to absorption in the visible. Sketch these functions.

If the absorption cross-sections for H^- and H bound-free opacity in the Paschen continuum are similar, estimate the temperature below which H^- is the most important source of continuous absorption. (Relevant ionization and excitation potentials are in the notes.)

8. An F star has a temperature of 7000 K. Microturbulence in the atmosphere has RMS velocity 3 km/s. Determine the full width at half maximum of an optically thin line of iron with wavelength 400 nm. You may have to consult your *Nebulae* notes for a reminder of how to combine microturbulent and thermal velocities in line broadening calculations.

9. The solution of the equation of radiation transfer for Eddington-Milne line formation was outlined in the notes. Derive it in detail, working through the moment equations, applying boundary conditions, and going through the algebra, to finally get the residual flux in a line. Then derive equations for the residual flux for the following situations:

Scattering lines:	no scattering in continuum, pure scattering in line
Absorption lines:	pure scattering in continuum, no scattering in line

Hint: what are the line and continuum ε values for each case?

What are the residual fluxes for very strong lines in the above cases? Comment on the spectral appearance of the line cores. For a very strong line, consider the limit of very large η.

10. In a Monte Carlo code that simulates emission and scattering in a spherical circumstellar shell, the radial dependence of the emissivity is $j(r) \propto (r/R)^{-\alpha}$, where r is in the range $R < r < R_{\max}$. Derive an expression for randomly sampling the radial location for emitting photons in the shell.

11. The Rayleigh scattering phase function is independent of azimuthal angle, ϕ, and has the dependence on polar angle, θ: $P(\theta) \propto 1 + \cos^2\theta$. What is the normalization factor so that the scattering phase function is normalized over all solid angles? How would you choose θ and ϕ values to randomly choose a scattering direction? Hint: you may not be able to derive analytic expressions for randomly choosing both θ and ϕ.