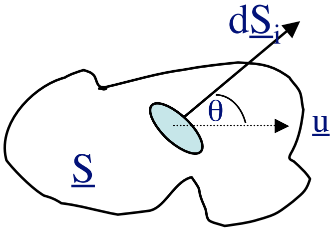
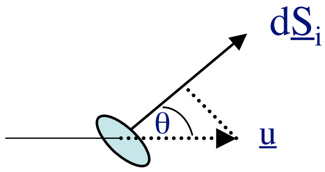
4. Example: flux of mass

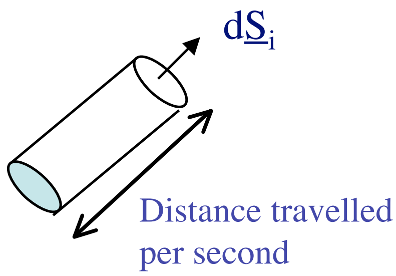
- Consider a volume V whose surface $\underline{S}$ is a patchwork of surface elements $d\underline{S}$



- A flow $\underline{u}$ through the surface has a component along the outward normal = u cosθ



- Every second:
  - this flow travels a distance u cosθ in the direction of dS
  - a mass of m = (density.vol) = ρ (u cosθ dS) flows through each surface element dS



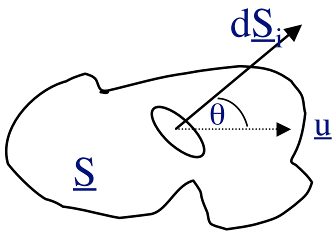
- Noting that $$u\cos\theta = \frac{\underline{u}\cdot d\underline{S}}{|d\underline{S}|}$$

- We can write this mass flux as $$m = \rho\underline{u}\cdot d\underline{S}$$

- The total rate at which mass (density ρ) flows through the surface S is the sum over all the elements dS:

$$-\sum_i \rho \underline{u}\cdot d\underline{S}_i = -\int_s \rho \underline{u}\cdot d\underline{S}$$

$$= -\int_V \underline{\nabla}\cdot(\rho \underline{u})dV$$



Because as drawn, it is an <u>out</u>flow

Divergence theorem

- In the absence of sources or sinks of mass, this must be equal to the rate of change of mass of the fluid in V, so that

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int \underline{\nabla}\cdot(\rho \underline{u})dV$$

$$\Rightarrow$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u})\right) dV = 0$$

- Since this must be true for all volumes,

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0} \qquad (4.1)$$

- This is the Eulerian form.

- The Lagrangian form is $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \underline{u} \cdot \underline{\nabla}\rho$$

- We can use (4.1) to write this as

$$\frac{d\rho}{dt} = -\underline{\nabla} \cdot (\rho \underline{u}) + \underline{u} \cdot \underline{\nabla}\rho$$

- And by expanding out the divergence:

$$\underline{\nabla} \cdot (\rho \underline{u}) = \rho \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla}\rho$$

- We get $$\frac{d\rho}{dt} = -\rho \underline{\nabla} \cdot \underline{u}$$

i.e. $$\boxed{\frac{d\rho}{dt} + \rho \underline{\nabla} \cdot \underline{u} = 0} \tag{4.2}$$

- NB: in incompressible fluids $\frac{d\rho}{dt} = 0$
- And so the flow is "divergence free"

$$\Rightarrow \boxed{\underline{\nabla} \cdot \underline{u} = 0} \tag{4.3}$$

Question 2

- If you have a steady, incompressible 2D flow $\underline{u}=(u_x,u_y)$, where $u_y=-\sinh(y)$
- Use the equation of mass conservation to get u_x.

Answer 2

- Remember *steady and incompressible*!

$$\underline{\nabla} \cdot \underline{u} = 0$$

$$\Rightarrow \frac{\partial u_x}{\partial x} = -\frac{\partial u_y}{\partial y}$$

$$\Rightarrow u_x = -\int \frac{\partial u_y}{\partial y} dx$$

$$\Rightarrow u_x = x\cosh(y) + c$$