10. Sound waves

- How does a disturbance propagate in a fluid?
- Start with a static equilibrium: pressure p_0 and density ρ_0 are uniform and velocity $\underline{u}$=0.
- Consider small perturbations:

$$p = p_0 + \delta p$$
$$\rho = \rho_0 + \delta\rho$$
$$\underline{u} = \delta\underline{u}$$

- How do the perturbations $\delta\rho$,δp and δu behave?

- They have to conserve mass:

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u}) = 0$$
$$\Rightarrow$$
$$\frac{\partial(\rho_0 + \delta\rho)}{\partial t} + \underline{\nabla}\cdot[(\rho_0 + \delta\rho)\delta\underline{u}] = 0$$
$$\Rightarrow \quad \frac{\partial\rho_0}{\partial t} = 0$$
$$and \quad \frac{\partial(\delta\rho)}{\partial t} + \rho_0\underline{\nabla}\cdot(\delta\underline{u}) = 0 \qquad (10.1)$$

- and momentum conservation (neglecting terms of order δ^2 or higher) is

$$\rho\frac{\partial \underline{u}}{\partial t}+\rho(\underline{u}\cdot\underline{\nabla})\underline{u}=-\underline{\nabla}p$$

$$\Rightarrow$$

$$\rho_0\frac{\partial(\delta\underline{u})}{\partial t}=-\underline{\nabla}(\delta p)=-\frac{dp}{d\rho}\underline{\nabla}(\delta\rho) \qquad (10.2)$$

- Using $\frac{\partial}{\partial t}(10.1)-\underline{\nabla}\cdot(10.2)$ gives

$$\frac{\partial^2(\delta\rho)}{\partial t^2}=\frac{dp}{d\rho}\underline{\nabla}^2(\delta\rho) \qquad (10.3)$$

- A wave equation!

- So the density perturbations $\delta\rho$ travel like a *wave*.

- In 1-D this has solution $\delta\rho = Ae^{i(kx-\omega t)}$

- Where $\dfrac{\omega^2}{k^2} = \dfrac{dp}{d\rho}$

- Since:

$$\frac{\partial^2}{\partial t^2}(\delta\rho) = (-i\omega)^2 Ae^{i(kx-\omega t)} = -\omega^2\delta\rho$$

$$\nabla^2(\delta\rho) = (ik)^2 Ae^{i(kx-\omega t)} = -k^2\delta\rho$$

- But ω/k is the phase velocity (i.e. speed of points at constant phase)
- Wave travels at

$$c_s = \sqrt{\frac{dp}{d\rho}}$$

- Sound waves propagate due to an interplay between density and pressure variations

- Density perturbation -> pressure gradient -> accelerations -> velocities -> density perturbations etc

- Sound speed depends on how pressure forces react to changes in density i.e $dp/d\rho$

- A stiff equation of state (high $dp/d\rho$) => large restoring force for small density perturbations=> rapid propagation.

- Note that the wave speed
  - is different from $\delta \underline{u}$ which is the velocity of the fluid elements…c_s is the *pattern speed* of the disturbance
  - is not a function of ω so sound waves propagate isotropically

Which equation of state?

- Isothermal : $dp/d\rho$ at constant T
  - Rarefactions and compressions have time (over a timescale $1/\omega$) to exchange heat and maintain a constant T.
  - $c_s^2 = k_B T/m$

- Adiabatic : $dp/d\rho$ at constant entropy s
  - No time for heat exchange, so compressions heat up and rarefactions cool (pdV work)
  - $c_s^2 = \gamma\, k_B T/m$

Note:

- Wave speed is only a function of T
- Efficiency of heat exchange (by conduction or radiation) determines whether adiabatic or isothermal.
- Perturbations don't have to behave like the ambient medium..can have adiabatic waves in an isothermal atmosphere