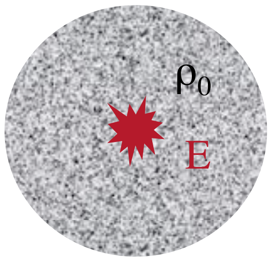
12. Explosion into a uniform medium

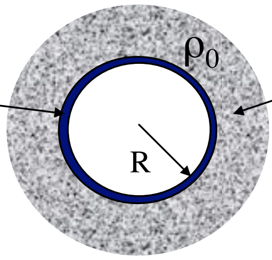
- Solution is characterised by two variables:
  - The explosion energy E, delivered instantaneously at a point
  - The density ρ_0 of the medium



- Note that we ignore the temperature of the medium (i.e. set T_0=0) so we don't consider the role of the thermal pressure of the medium confining the explosion.

- As explosion propagates out, a shock forms and the medium is swept up into a thin shell of shocked gas.

Shocked layer, thickness D, density ρ_1



Undisturbed medium, density ρ_0

- Since the shock is strong we have (for an adiabatic shock)

$$T_0 \to 0 \Rightarrow M \to \infty$$

$$\frac{\rho_0}{\rho_1} = \frac{\gamma - 1}{\gamma + 1}$$

- If all the mass is swept into this thin layer

$$\frac{4\pi}{3}\rho_0 R^3 = 4\pi\rho_1 R^2 D$$

$$\Rightarrow D = \frac{1}{3}\frac{\gamma - 1}{\gamma + 1} R \quad \text{or} \quad D/R \approx 0.08 << 1 \ \text{for} \ \gamma = 5/3$$

- Remember RH1:

$$\rho_0 u_0 = \rho_1 u_1$$

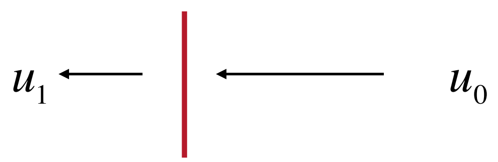
- So for a strong shock

$$u_1 = \frac{\rho_0}{\rho_1} u_0 = \frac{\gamma - 1}{\gamma + 1} u_0$$

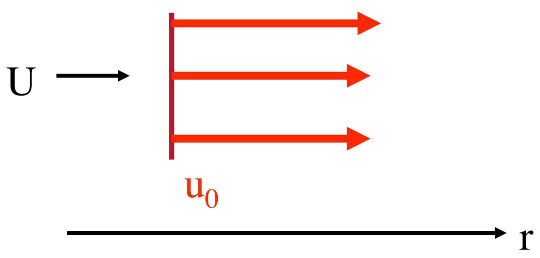
- For γ=5/3

$$u_1 = \frac{1}{4} u_0$$

- In the shock frame:



- In a rest frame



- So relative to the unshocked gas, the velocity of the shocked gas U is

$$U = u_0 - u_1 = \frac{2u_0}{\gamma + 1}$$

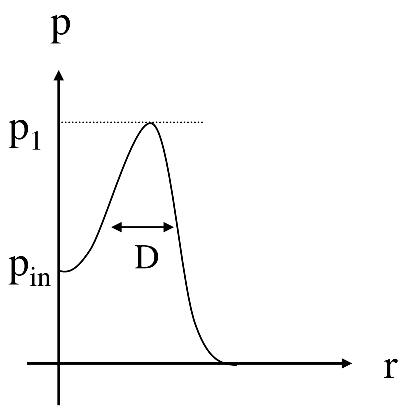
- As the shell grows it gains momentum at a rate:

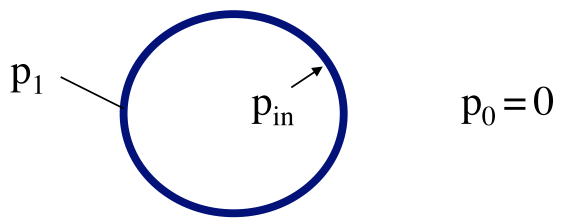
$$\text{Rate} = \frac{d}{dt}\left[\underline{\frac{4\pi}{3}\rho_0 R^3}\ \underline{\frac{2u_0}{\gamma + 1}}\right]$$

mass of gas swept up

velocity of gas swept up

- This momentum gain is provided by the pressure acting on the inside of the shell, p_{in}
  - $p_{in} \neq p_1$
  - $T_0 = 0$ => no opposing counter pressure from the unshocked medium.





So what is p_{in}?

- Set

$$p_{in} = \alpha p_1$$

constant

- For a strong shock

$$p_1 = \frac{2}{\gamma + 1} \rho_0 u_0^2$$

- And so the momentum equation (force = rate of change of momentum) becomes

$$4\pi R^2 \alpha \underbrace{\frac{2}{\gamma+1}\rho_0 u_0^2}_{p_{in}} = \frac{d}{dt}\left[\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma+1}\right]$$

- Re-arrange to get

$$3\alpha R^2 u_0^2 = \frac{d}{dt}\left[R^3 u_0\right]$$

- But u_0 is the speed with which the shock advances on the unshocked gas, i.e.

$$u_0 = \frac{dR}{dt}$$

$$3\alpha R^2 \dot{R}^2 = \frac{d}{dt}\left[R^3 \dot{R}\right]$$

- Try

$$R \propto t^b \Rightarrow b = \frac{1}{4 - 3\alpha}$$

- We now have a shell that grows with time as

$$\left.\begin{aligned} R &\propto t^{\frac{1}{4-3\alpha}} \\ u_0 &\propto t^{\frac{3\alpha-3}{4-3\alpha}} \end{aligned}\right\} \qquad (12.1)$$

- But we still don't know α!

- Use the explosion energy E which for an adiabatic blast wave must be conserved.

- E goes into two forms:

  – The kinetic energy of the shell:

$$\frac{1}{2}\frac{4\pi}{3}\rho_0 R^3 U^2$$

  – The internal energy

- Internal energy per unit volume is p/(γ-1).

- Most of the volume of the bubble created by the blast wave is in the internal cavity, so assume that ALL the internal energy is in the cavity.

- The amount of internal energy is then

$$\frac{4\pi}{3}R^3 \underline{\frac{\alpha p_1}{\gamma - 1}}$$

Internal energy/unit volume

- So

$$E = \frac{4\pi}{3}R^3\left[\frac{1}{2}\rho_0 U^2 + \frac{\alpha}{\gamma - 1}\frac{2\rho_0 u_0^2}{\gamma + 1}\right]$$

$$U^2 = \left(\frac{2u_0}{\gamma + 1}\right)^2$$

- hence

$$E = \frac{4\pi}{3}R^3\frac{\rho_0 u_0^2}{\gamma + 1}\left[\frac{2}{\gamma + 1} + \frac{2\alpha}{\gamma - 1}\right] \qquad (12.2)$$

- So, using (12.1)

$$E \propto R^3 u_0^2 \propto t^{\frac{6\alpha - 3}{4 - 3\alpha}}$$

- But E is conserved, so

$$6\alpha - 3 = 0 \Rightarrow \alpha = 1/2$$

- And we get

$$R \propto t^{2/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}$$

- The maximum radius R_{max} of the blast wave is reached when
  - the pressure behind the shock is similar to that ahead of the shock.
  - At this point the blast wave is no longer a shock, but just a compression wave

- Estimate R_{max} by putting

$$p_1 \approx p_0 = \frac{\rho_0 c_{s0}^2}{\gamma} = \frac{2}{\gamma + 1}\rho_0 u_0^2$$

$$\Rightarrow u_0^2 \approx \frac{(\gamma + 1)c_{s0}^2}{2\gamma}$$

- But from the energy equation (12.2) with α=1/2 we have

$$u_0^2 = \frac{(\gamma + 1)^2(\gamma - 1)}{(3\gamma - 1)} \frac{3E}{4\pi\rho_0 R^3}$$

- Equating these we have

$$E \approx \frac{(3\gamma - 1)}{2(\gamma + 1)} \frac{4\pi R_{max}^3}{3} \frac{\rho_0 c_{s0}^2}{\gamma(\gamma - 1)}$$

$$\frac{p_0}{(\gamma - 1)}$$

Thermal energy originally contained within a sphere of radius R_{max}

Thermal energy per unit volume in undisturbed gas

- So the criterion $p_1 \sim p_0$ is approximately
  - $u_0 \sim c_{s0}$ (transition to subsonic flow)
  - Blast wave reaches the radius where the explosion energy = original thermal energy of the original gas contained in that sphere
  - NB: criteria are approximate because there is no sudden transition between the blast wave and sound wave behaviour.