Lecture 4: Building a model - metrics

1. Our axioms: The Cosmological Principle
2. What is a metric
3. A curved metric
4. An expanding metric
5. A 3D curved expanding metric:  
   - The Robertson-Walker metric
6. Simplification:  
   - Radial paths  
   - Photons
7. Meaning of Redshift
8. Distances in Modern Cosmology  
   - Comoving & Proper distance  
   - Angular diameter distance  
   - Luminosity distance  
   - Surface Brightness

Course Text: Chapter 5
Wikipedia: Metric Expansion of Space, FRW, Comoving distance
The Cosmological principle

- **Axiom:**
  "The Universe is both homogeneous and isotropic"

  Same in all locations
  Same in all directions
  Laws of Physics Universal

- **Supporting evidence:**
  Cosmic Microwave Background
  Deep galaxy counts
  Large scale galaxy surveys
Homogeneity and Isotropy

homogeneous
not isotropic

isotropic
not homogeneous

Universal expansion is the only motion allowed by Cosmological Principle
The Standard Cosmological Model

A metric system for defining the geometry = spacetime-metric

A census of the key contents= vacuum energy, matter, radiation

A model for how the contents modify the geometry = gen. rel.

Observational Cosmology:

Can measure geometry and use model to constrain contents:
- Standard candles
- Baryon Acoustic Oscillations
- CMB Anisotropy Peak

Can obtain direct measurements of contents (matter, radiation):
- Mass density via nearby studies (galaxy surveys)
- Radiation density from CMB BB spectrum

Can use orthogonal evidence:
- Age of galaxy, stars, Earth < Age of Universe
- Abundances via Big Bang Nucleosynthesis
Curved Geometries

- Curved geometries in n-dimensional space equate to surfaces of objects in (n+1)-dimensional space.

E.g., 1D FLAT

Need to visualise curved 1D in 2D space
• 2D curved surfaces

• GR treats mass as a geometric effect in 3D space. We therefore need to go to 4D to visualise it, or define it mathematically.
Is our Universe Curved?

Closed

- Curvature: +
- Sum of angles of triangle: > 180°
- Circumference of circle: < 2πr
- Parallel lines: converge
- Size: finite
- Edge: no

Flat

- Curvature: 0
- Sum of angles of triangle: = 180°
- Circumference of circle: = 2πr
- Parallel lines: remain parallel
- Size: infinite
- Edge: no

Open

- Curvature: --
- Sum of angles of triangle: < 180°
- Circumference of circle: > 2πr
- Parallel lines: diverge
- Size: infinite
- Edge: no
Flat Space: Euclidean Geometry

Cartesian coordinates:

1 D: \( dl^2 = dx^2 \)
2 D: \( dl^2 = dx^2 + dy^2 \)
3 D: \( dl^2 = dx^2 + dy^2 + dz^2 \)
4 D: \( dl^2 = dw^2 + dx^2 + dy^2 + dz^2 \)

\( dl^2 \) is the METRIC for whichever dimension you want to work in, just an expression that determines the line distance between two objects.
**Polar Coordinates**

*ND Polar coordinates: Often more useful as it's easier to measure a radial distance and angular separation than Cartesian distances*

**Easier to measure** $dr$ & $d\theta$ than $dx$ & $dy$

Radial coordinate $r$, angles $\phi, \theta, \alpha, ...$

1 D:  

$$dl^2 = dr^2$$

2 D:  

$$dl^2 = dr^2 + r^2 \ d\theta^2$$

3 D:  

$$dl^2 = dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \ d\phi^2 \right)$$

4 D:  

$$dl^2 = dr^2 + r^2 \left[ d\theta^2 + \sin^2 \theta \left( d\phi^2 + \sin^2 \phi \ d\alpha^2 \right) \right]$$

$$dl^2 = dr^2 + r^2 \ d\psi^2 \quad \text{generic angle: } \ d\psi^2 = d\theta^2 + \sin^2 \theta \ d\phi^2 + ...$$
Using the Metric

A metric allows us to define separation between two points in n-dimensional space:

\[ dl^2 = dx^2 + dy^2 + dz^2 \]
\[ dl^2 = dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

Also allows us to define geometric relationships:

**E.g., 2D**

\[ dl^2 = dr^2 + r^2 d\theta^2 \]
\[ A = \int_0^{2\pi} \int_0^r r \, dr \, d\theta = \pi r^2 \]
\[ C = r \int_0^{2\pi} d\theta = 2\pi r \]

**E.g., 3D**

\[ V = dxdydz \]
\[ V = \int_0^{2\pi} \int_0^r \int_0^{\pi/2} r^2 \, dr \, d\theta \sin \theta \, d\phi = \frac{4}{3} \pi r^3 \]
\[ A = r \int_0^{2\pi} \int_0^{-\pi/2} \int_0^r d\theta \sin \theta \, d\phi = 4\pi r^2 \]
Embedded Spheres

We want to define a metric for a 3D surface which could be curved. We therefore need to define the surface of a 4D-sphere

1 – D : \( R^2 = x^2 \) \hspace{1cm} 0 - D \hspace{0.5cm} 2 \text{ points}

2 – D : \( R^2 = x^2 + y^2 \) \hspace{1cm} 1 - D \hspace{0.5cm} \text{circle}

3 – D : \( R^2 = x^2 + y^2 + z^2 \) \hspace{1cm} 2 - D \hspace{0.5cm} \text{surface of 3 - sphere}

4 – D : \( R^2 = x^2 + y^2 + z^2 + w^2 \) \hspace{1cm} 3 - D \hspace{0.5cm} \text{surface of 4 - sphere}

\( R = \text{radius of curvature} \)
4D Cartesian metric is:

\[ dl^2 = dx^2 + dy^2 + dz^2 + dw^2 \]

Let: \( r^2 = x^2 + y^2 + z^2 \)

\[ w^2 = R^2 - r^2 \]

\[ \Rightarrow 2wdw = -2rdr \]

Now want to eliminate \( dw^2 \):

\[
(dw^2) = \left( \frac{rdr}{R^2 - r^2} \right)^2 = \frac{r^2 dr^2}{R^2 - r^2}
\]

\[ dl^2 = dx^2 + dy^2 + dz^2 + \frac{r^2 dr^2}{R^2 - r^2} \]
Usual Cartesian to Polar Conversion:

\[ z = r \cos \theta, \quad x = r \sin \theta \cos \phi, \quad z = r \sin \theta \sin \phi \]

\[ \Rightarrow \]

\[ dl^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r^2 dr^2}{R^2 - r^2} \]

\[ dl^2 = dr^2 \left[ 1 + \frac{1}{R^2/r^2 - 1} \right] + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ dl^2 = \frac{dr^2}{1-r^2/R^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

For curvature in the other direction one gets + so can introduce k

\[ dl^2 = \frac{dr^2}{1-k(r^2/R^2)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

k=-1=OPEN, k=0=FLAT, k=1=CLOSED

This expression defines metrics along a 3D curved surface, it does not yet include any expansion/contraction with time. Note in some texts R is set to 1
Note, sometimes see:

\[ r = R \sin \chi, \quad r = \text{distance to object}, \quad R = \text{radius of sphere}, \quad \chi = \text{opening angle} \]

\[ dr = R \cos \chi \, d\chi \]

\[ dl^2 = \frac{R^2 \cos^2 \chi \, d\chi}{1 - \sin^2 \chi} + R^2 \sin^2 \chi(\ldots) \]

\[ dl^2 = R^2 (d\chi + \sin^2 \chi(\ldots)) \]

We now have a 3D spatial metric which allows for curvature but does not yet include and expansion to do this we need to add in time.

Crude 1\textsuperscript{st} order estimates tell us light travel time is important.

We need our metric to connect “When” with “Where”

i.e., SPACE+TIME→SPACETIME
At $T=1$ two objects can be separated by $dl^2$ and at some time later by $R(t)^2dl^2$ where $R(t)$ is a SCALE FACTOR (i.e., a yardstick that expands with the Universe against which invariant distances can be calibrated).

$r_1 = \text{INVARİANT}$

$R_0 = R(T=0)$

$R_1 = R(T=1)$
Spacetime

- We can choose an arbitrary time to convert all distances too = TODAY
  - Proper Distance or Co-moving distance
  - In previous example $R_0 r_1$ is the co-moving distance.

- The cosmological principle only allows a uniform expansion in all spatial dimensions, so must act on our entire metric.

  \[ d\sigma^2 = R_0^2 dl^2, \text{ or more generally: } d\sigma^2 = R(t)^2 dl^2 \]

- 2 problems:
  - What is $R(t)$ [Next Lecture]
  - This metric can only describe simultaneous events, albeit simultaneous events at any time.
Connecting non-simultaneous events

- So now we have:
  \[ d\sigma^2 = R(t)^2 \left[ \frac{dr^2}{1 - k\left(\frac{r^2}{R^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \]

- But how to connect non-simultaneous events?

- Consider two non-simultaneous events occurring at the same location, the metric will then simply be the time interval.

\[ ds^2 = c^2 dt^2 \]

SPACETIME METRIC

Time interval multiplied by c
To give units of distance.
• How about: \[ ds^2 = c^2 dt^2 + d\sigma^2 \]?

• This just treats time as a 4\textsuperscript{th} dimension, i.e.,

\[ ds^2 = c^2 dt^2 + dx^2 + dy^2 + dz^2 \]

• But time is clear NOT a spatial dimension we cannot freely move back and forth.

• Special relativity, and the constancy of c provides an alternative possibility, consider:
  
  – two Fundamental Observers in different inertial frames who happen to occupy the same location at time T=0

\[ \rightarrow v \]

  – at this instant a large bomb is detonated.
• FO1 sees:
  – F01 inertial frame:

• FO2 sees:
  – F02 inertial frame:

A sphere of radiation moving Out with speed $c$ which can be expressed as:

$$R^2 = c^2 dt^2 = dx^2 + dy^2 + dz^2$$

Note $c$ is invariant (Spec. Rel)

$$R^2 = c^2 dt'^2 = dx'^2 + dy'^2 + dz'^2$$

It follows:

$$0 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$0 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$
Robertson-Walker metric

\[ dl^2 = \frac{dr^2}{1 - k\left(\frac{r}{R}\right)^2} + r^2 d\psi^2 \]

But events can be separated in time as well as space

\[ ds^2 = c^2 dt^2 - dl^2 \]

This is a space-time interval and defined so the path of a photon is always \( ds^2 = 0 \)

We also need to allow for the spatial dimensions to expand as some function of time.

\[ ds^2 = c^2 dt^2 - R(t)^2 dl^2 \]
The Robertson-Walker metric

\[ ds^2 = c^2 dt^2 - R^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

Space-time interval/metric

Normal spatial polar coords

Light travel distance

Curvature Term (open, closed, flat)

Expansion factor

This metric is forced by the Cosmological Principle (universal expansion)

Spacetime interval for a photon = 0 (from special relativity)

If we consider photons only on radial paths towards us in a flat Universe (k=0):

\[ c^2 dt^2 = R^2(t) dr^2 \]
Proper distance

\[ c^2 dt^2 = R(t)^2 \left( \frac{dr^2}{1 - kr^2} \right) \]

\[ \Rightarrow \]

\[ \int_0^{r_i} \frac{dr}{(1 - kr^2)^{1/2}} = c \int \frac{dt}{R(t)} \]

For k=0:

\[ r_i = c \int_0^{t_i} \frac{dt}{R(t)} \]

\[ \therefore \]

Proper distance = \( R_0 r_i = cR_0 \int_0^{t_i} \frac{dt}{R(t)} \)

If radiation connects us to a distant object we can express its proper distance in terms of \( R(t) \) and the time interval.

I.e., DISTANCE INTERVAL = TIME INTERVAL

Time to connect to observations
Redshift

- Imagine a distance source which emits light of period $T$ (i.e., $\nu = 1/T$)

- Definition of redshift:
- Or using our metric:

$$\frac{\lambda_{\text{received}}}{\lambda_{\text{emitted}}} = \frac{T_{\text{received}}}{T_{\text{emitted}}} = (1 + z)$$
Redshift

$$\int_{0}^{r_1} \frac{dr}{(1 - kr^2)^{1/2}} = c \int_{t_0}^{t_0 + dt_0} \frac{dt}{R(t)} \quad \leftarrow \quad r_1 = \text{invariant length of one wavelength}$$

If: \( R(t_0) \approx R(t_0 + dt_0) \)

$$\Rightarrow \int_{0}^{r_1} \frac{dr}{(1 - kr^2)^{1/2}} = \frac{c}{R_0} T_{\text{received}} \quad \leftarrow \quad dt_o = T_{\text{received}}$$

also at source: \( dt_1 = T_{\text{emitted}} \)

$$\Rightarrow \int_{0}^{r_1} \frac{dr}{(1 - kr^2)^{1/2}} = \frac{c}{R_1} T_{\text{emitted}}$$

combining:

$$\frac{c}{R_0} T_{\text{received}} = \frac{c}{R_1} T_{\text{emitted}}$$

$$\Rightarrow \frac{\lambda_{\text{received}}}{\lambda_{\text{emitted}}} = \frac{T_{\text{received}}}{T_{\text{emitted}}} = \frac{R_0}{R_1} = (1 + z_1)$$

Generalising:

$$\frac{R_0}{R(t)} = (1 + z)$$

Expansion \( \rightarrow z > 0 \) redshift

Contraction \( \rightarrow z < 0 \) redshift
Redshift

- An expansion implies a stretching of space-time.
- The more space-time there is between you and an object the faster it will appear to be moving away.
- It is the expansion which causes a galaxy’s spectrum to be REDSHIFTED:

REDSHIFT IS NOT THE SAME AS DOPPLER SHIFT
Cosmological Distances

- **Proper distance** \( (d_p) \)
  - Actual distance to object today (if one could stop time and measure it with a ruler).

- **Angular diameter distance** \( (d_a) \)
  - Proper distance at time of photon emission.

- **Luminosity distance** \( (d_l) \)
  - Distance the photon feels it has travelled.

\[
d_p = (1 + z) d_a = \frac{d_l}{(1 + z)}
\]

Why?
Proper distance

- **co-moving radial distance**
  - now (when photon received):
    \[ d_{z=0} = R(t_0) r_o = R_0 r_o \]
  - when photon emitted:
    \[ d_z = R(t_e) r_o = \frac{R(t_e)}{R_0} R_0 r_o = \frac{d_{z=0}}{1 + z} \]
    \[
    \left[ (1 + z) = \frac{\lambda_{\text{observed}}}{\lambda_{\text{laboratory}}} = \frac{R(t_o)}{R(t_e)} \right]
    \]

- **Proper distance (for flat universe):** \[ d_p = d_{z=0} = R_o r_o \]
- **Angular diameter distance:**
  \[ d_a = r_e = \frac{d_p}{1 + z} \]
Angular diameter distance

Angles frozen at emission. If structure bound need to use distance at emission to convert angular sizes to physical sizes.

\[ r = d_a \tan(\theta) \]

For non-bound objects use proper distances to work out tangential sizes.
Luminosity Distance

- Luminosity ( erg s\(^{-1}\) )
  \[ L = \frac{N \ h \ \nu_e}{\Delta t_e} \]

- area of photon sphere
  \[ A = 4\pi \ d_P^2 \]

- redshift:
  \[ \lambda_o = \lambda_e \ (1 + z) \]
  \[ \nu_o = \nu_e / (1 + z) \]

- time dilation: lower photon arrival rate
  \[ \Delta t_o = \Delta t_e (1 + z) \]

- observed flux ( erg cm\(^{-2}\) s\(^{-1}\) )
  \[ F = \frac{N \ h \ \nu_o}{A \ \Delta t_o} = \frac{L}{4\pi \ d_P^2 \ (1 + z)^2} = \frac{L}{4\pi \ d_L^2} \]

- Luminosity distance
  \[ d_L = d_P (1 + z) \]

Sources look fainter.
Surface Brightness

- **Solid angle**
  \[ \Omega = \frac{A}{D_A^2} \]

- **Surface brightness**
  - Flux per solid angle (erg s\(^{-1}\) cm\(^{-2}\) arcsec\(^{-2}\))
  \[ \Sigma \equiv \frac{F}{\Omega} = \frac{L}{4\pi D_L^2} \frac{D_A^2}{A} = \frac{L}{4\pi A (1 + z)^4} \]
  - decreases very rapidly with z because:
    - expansion spreads out the photons
    - decreases their energy
    - decreases their arrival rate
Distance summary

• Proper distance (actual distance today)

\[ d_p = R_0 r_1 \]

• Angular diameter distance (for calculating bound sizes)

\[ d_a = \frac{d_p}{(1 + z)} \]

• Luminosity distance (for deriving fluexs)

\[ d_l = (1 + z)d_p \]
1st Assignment

• Calculate the primordial abundances (X_\text{p} & Y_\text{p}) assuming:
  – The photon-to-baryon ratio today is 1:1
  – The photon-to-baryon ratio today is 1:10^{10}

• Use course notes available on WebCT

• Please hand in neat solutions with explanation of your reasoning by 2\textsuperscript{nd} May 11am (i.e., next lecture)

There will be three marked assignments which carry 40\% of the mark for this course component.

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End of lecture 4
Distant objects are seen as old objects because of the light travel time.

Our metric needs to accommodate this.